Class Notes on General Equilibrium Theory
Revealed preference tests of utility maximization and general equilibrium

Let $\mathcal{O} = \{(p_t, x_t)\}_{t \in T}$ be a set of observations, where $x_t \in R^l_+$ is the observed demand at price $p_t \in R^l_{++}$.

We know that certain observations are *not* consistent with utility-maximization with increasing utility functions. In other words, the hypothesis leads to *observable restrictions* on a data set.

What restrictions on the data are sufficient for us to recover a utility function generating the data?

The set $\mathcal{O}$ satisfies the generalized axiom of revealed preference (GARP) if whenever there are observations satisfying ($\star$):

$$\begin{aligned} p_{t^1} \cdot x_{t^2} &\leq p_{t^1} \cdot x_{t^1} \\ p^{t^2} \cdot x_{t^3} &\leq p_{t^2} \cdot x_{t^2} \\ &\cdot \\ &\cdot \\ p_{t^{n-1}} \cdot x_{t^n} &\leq p_{t^{n-1}} \cdot x_{t^{n-1}} \\ p_{t^n} \cdot x_{t^1} &\leq p_{t^n} \cdot x_{t^n}, \end{aligned}$$

then all the inequalities have to be equalities.

Proposition: *Suppose that $\mathcal{O}$ is drawn from a consumer with an increasing utility function. Then it must satisfy GARP.*

Proof: Suppose that $\mathcal{O}$ is drawn from a consumer maximizing the utility function U. Note that $p_s \cdot x_r \leq p_s \cdot x_s$ means that $U(x_r) \leq U(x_s)$. Since U is increasing, if $p_s \cdot x_r < p_s \cdot x_s$ then $U(x_r) < U(x_s)$. The displayed inequalities in ($\star$) imply that

$$U(x_{t_1}) \leq U(x_{t_n}) \leq U(x_{t_{n-1}}) \leq ... \leq U(x_{t_3}) \leq U(x_{t_2}) \leq U(x_{t_1}). \qquad (1)$$

Clearly we obtain a contradiction if any inequality in (1) is strict. So they must all be equalities, which mean that the observations satisfying ($\star$) must also be equalities. **QED**

What is really striking is that this result has a converse, so that GARP exhausts *all* the observable implications of utility maximization with an increasing utility function.

Theorem: (Afriat) *Let $\mathcal{O} = \{p_t, x_t\}_{t\in T}$ be a set of observations. The following statements about $\mathcal{O}$ are equivalent:*

(1) There is an strictly *increasing and concave utility function U such that*

$$x_s \in \text{argmax}_{x\in B(p_s, p_s\cdot x_s)} U(x).$$

(2) $\mathcal{O}$ obeys GARP.

(3) There are numbers ϕ_s and λ_s, with $\lambda_s > 0$ (associated to each observation $s \in T$) such that

$$\phi_s \leq \phi_k + \lambda_k p_k \cdot (x_s - x_k) \text{ for any observations } s \text{ and } k. \tag{2}$$

The problem of finding (ϕ_s, λ_s) (for each observation s) that solve (2) is a linear programming problem. Such problems are known to be *solvable* in the sense that there is an algorithm to determine in a finite number of steps, whether or not the system of linear inequalities given by (2) has a solution.

The first proposition says that whenever U is increasing, $\mathcal{O}$ obeys GARP. Afriat's Theorem also says that if GARP holds, then there is a utility function generating $\mathcal{O}$ that is increasing *and concave*. Hence, if a data set is consistent with an increasing utility function, it must be consistent with an increasing and concave utility function.

In fact, the utility function constructed from the data is

$$U(x) = \min_{(p_t,x_t)\in\mathcal{O}} \{\phi_t + \lambda_t p_t \cdot (x - x_t)\}$$

(with ϕ_t and λ_t satisfying (2)). This is always a concave function, and it is an increasing utility function since $\lambda_t > 0$ for all t.

Proof of Afriat's Theorem: The first proposition says that (1) implies (2).

To see that (3) implies (1) consider the utility function $U : R^l_+ \to R$ given by

$$U(x) = \min_{(p_t, x_t) \in \mathcal{O}} \{\phi_t + \lambda_t p_t \cdot (x - x_t)\}.$$

This is an increasing utility function since $\lambda_t > 0$ for all t. We claim that it generates the observations in $\mathcal{O}$.

Let x satisfy $p_s \cdot x = p_s \cdot x_s$, i.e., x is on the budget plane of the agent at price p_s and income $p_s \cdot x_s$. It follows from the definition of U that $U(x) \leq \phi_s$. To guarantee that x_s maximizes utility in $B(p_s, p_s \cdot x_s)$ it suffices to show that $U(x_s) = \phi_s$. This follows immediately from (2).

It remains for us to show that (2) implies (3). This is the hard part of Afriat's Theorem. Instead of proving the full result, we simplify the problem by assuming that the set of observations is *generic* in the sense that for any $t' \neq t$, $p_t \cdot x_{t'} \neq p_t \cdot x_t$. In other words, either $p_t \cdot x_{t'} < p_t \cdot x_t$ or $p_t \cdot x_{t'} > p_t \cdot x_t$.

Proof that (2) implies (3), assuming that observations are generic: Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x_t\}_{t \in T}$. The order $\succ^{**}$ on $\mathcal{X}$ is defined as follows: $x^t \succ^{**} x_s$ if $p_t \cdot x_s < p_t \cdot x_t$. In this case we say that x^t is *directly revealed preferred* to x^s.

We denote with $\succ^*$ the *transitive closure* of $\succ^{**}$, i.e., $x_t \succ^* x_s$ if there are observations t^1, t^2,..., t^n such that

$$x_t \succ^{**} x_{t_1} \succ^{**} x_{t_2} \succ^{**} \ldots \succ^{**} x_{t_n} \succ^{**} x_s.$$

If $x_t \succ^* x_s$ we say that x_t is *revealed preferred* to x_s, and refer to $\succ^*$ as *the revealed preference order on* $\mathcal{X}$ derived from $\mathcal{O}$.

1
partial

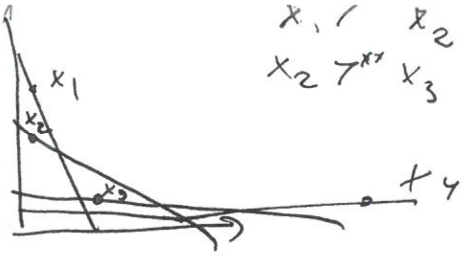
Lemma: *The order $\succ^*$ is transitive and irreflexive.*

Proof: That it is transitive follows almost immediately from its definition. The order is irreflexive if for any $x_t \in X$, $x_t \not\succ^* x_t$. This is true because there are no cycles, i.e., there does not exist t^1, t^2,..., t^n such that

$$x_{t^1} \succ^* x_{t^2} \succ^* .... \succ^* x_{t^n} \succ^* x_{t^1}.$$

Such a cycle will violate GARP. **QED**

An order $\succ$ on $\mathcal{X}$ is an *extension* of another order $\succ^*$ if $x_t \succ x_s$ whenever $x_t \succ^* x_s$.



$x_1 \,/\, x_2$ $x_1 >^* x_2$ so: $x_1 >^* x_2 >^* x_3$
$x_2 >^{**} x_3$ $x_1 >^* x_3$
x_4?
Extend: (many)
$x_4 > x_1 > x_2 > x_3$

Lemma: *There is an extension $\succ$ of $\succ^*$ that is transitive, irreflexive, and complete, i.e., for any two elements x_t and x_s in $\mathcal{X}$, either $x_t \succ x_s$ or $x_s \succ x_t$.*[1]

Proof: We prove this by induction on the number of elements in the set of observations. If this set has just two elements, the claim is obviously true. Suppose it is true for any set with $N-1$ observations and assume that $\mathcal{O}$ has N observations. Choose an element $x_{\bar{t}}$ such that there is no $x_s \in \mathcal{X}$ with $x_s \succ^* x_{\bar{t}}$. We claim that such an element exists. First choose any $x_s \in \mathcal{X}$. If that is dominated by x_{t^1} choose x_{t^1}, if x_{t^1} is dominated by x_{t^2} choose x_{t^2}, and so on. This has to stop since $\mathcal{X}$ has finitely many elements and there are no cycles.

The set $\mathcal{O}' \equiv \mathcal{O} \setminus \{(p_{\bar{t}}, x_{\bar{t}})\}$ has $N-1$ elements. We denote $\mathcal{X}' = \mathcal{X} \setminus \{x_{\bar{t}}\}$. On $\mathcal{X}'$, we can define its revealed preference order $\succ^{*\prime}$ and there is a complete order $\succ'$ on $\mathcal{X}'$ extending $\succ^{*\prime}$.[2]

It is straightforward to check that for x_t and x_s in $\mathcal{O}'$, $x_t \succ^* x_s$ if and only if $x_t \succ^{*\prime} x_s$. (Note that the 'only if' part of this claim makes crucial use of the fact that $x_{\bar{t}}$ is undominated.) Define $\succ$ as the extension of $\succ'$ and with $x_{\bar{t}} \succ x_s$ for all $x_s \in \mathcal{O}'$. It is clear that $\succ$ completes $\succ^*$.

QED

Lemma: *Let $\succ$ be a complete extension of the revealed preference order $\succ^*$ of $\mathcal{X}$. By relabeling the entries if necessary, assume that $\mathcal{X} = \{x_1, x_2, ..., x_N\}$, with*

$$x_1 \prec x_2 \prec x_3 \prec ... \prec x_N.$$

Then there are numbers ϕ_s and λ_s, for $s = 1, 2, ..., N$, with $\lambda_s > 0$ for all s, such that

$$\phi_s < \phi_k + \lambda_k p_k \cdot (x_s - x_k) \text{ for all } k \neq s$$

(as required by Afriat's Theorem).

Proof: Denote $p_i \cdot (x_j - x_i)$ by a_{ij}. Choose ϕ_1 to be any number and λ_1 to be any *positive* number. Since $x_j \succ x_1$ for all $j > 1$,

$$a_{1j} = p_1 \cdot (x_j - x_1) > 0.$$

So there is ϕ_2 such that

$$\phi_1 < \phi_2 < \min_{j>1}\{\phi_1 + \lambda_1 a_{1j}\}.$$

Now choose $\lambda_2 > 0$ sufficiently small so that

$$\phi_1 < \phi_2 + \lambda_2 a_{21}.$$

[1] There could be more than one such extension.

[2] Note that $\succ^{*\prime}$ is defined with respect to $\mathcal{O}'$.

Since

$$a_{2j} = p_2 \cdot (x_j - x_2) > 0 \text{ for all } j > 2,$$

$\phi_2 < \min_{j>2}\{\phi_2 + \lambda_2 a_{2j}\}$. Therefore, we can choose ϕ_3 such that

$$\phi_2 < \phi_3 < \min\left\{\min_{j>2}\{\phi_2 + \lambda_2 a_{2j}\}, \min_{j>1}\{\phi_1 + \lambda_1 a_{1j}\}\right\}.$$

We can choose $\lambda_3 > 0$ and sufficiently small such that

$$\phi_i < \phi_3 + \lambda_3 a_{3i} \text{ for } i = 1, 2.$$

More generally, we can choose ϕ_k such that

$$\phi_{k-1} < \phi_k < \min_{s \leq k-1}\left\{\min_{j>s}\{\phi_s + \lambda_s a_{sj}\}\right\}$$

and $\lambda_k > 0$ and sufficiently small so that

$$\phi_i < \phi_k + \lambda_k a_{ki} \text{ for } i \leq k-1.$$

In this way, we obtain

$$\phi_s < \phi_k + \lambda_k a_{ks} \text{ for all } k \neq s,$$

as required. **QED**