A Tarski-Kantorovich Theorem for Correspondences[*]

Łukasz Balbus[†] Wojciech Olszewski[‡] Kevin Reffett[§]
Łukasz Woźny[¶]

December, 2023

Abstract

For a strong set order increasing (resp., strongly increasing) upper order hemicontinuous correspondence $F : A \rightrightarrows A$, where A is a complete lattice (resp., a σ-complete lattice), we provide tight fixed-point bound for sufficiently large iterations $F^k(a^0)$, starting from any point $a^0 \in A$. Our results, hence, prove a generalization of the Tarski-Kantorovich theorem. We provide an application of our results to a class of social learning models on networks.

Keywords: iterations of monotone correspondences; Tarski's fixed-point theorem; Veinott-Zhou version of Tarski's theorem for correspondences; Tarski-Kantorovich theorem for correspondences; adaptive learning.

JEL classification: C62, C65, C72

[*] We would like to thank John Quah for his comments on the earlier draft of this paper. Special thanks to Ben Golub for suggesting the application we analyze in Section 4. Łukasz Woźny acknowledges financial support by the National Science Center, Poland: NCN grant number UMO-2019/35/B/HS4/00346. Reffett acknowledges the Dean's Summer Research Award at the WP Carey School of Business at ASU for its financial support.

[†] Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Góra, Poland.

[‡] Department of Economics, Northwestern University, USA.

[§] Department of Economics, Arizona State University, USA.

[¶] Department of Quantitative Economics, SGH Warsaw School of Economics, Poland.

1 Introduction

The celebrated Tarski (1955)[1] fixed-point theorem has found numerous applications in various disciplines, including economics. The theorem states that an increasing transformation of a complete lattice has a complete lattice of fixed points. This result has been extended to the case of monotone correspondences in the work of Veinott (1992) and Zhou (1994).[2] In the case of Tarski's original fixed-point theorem, the lowest fixed point is the "limit" of the sequence of iterations starting from the lowest element of the lattice, and the highest fixed point is the "limit" of the sequence of iterations starting from the highest element of the lattice.[3] In his two recent papers, Olszewski (2021a,b) characterized the elements of the lattice that are the sharp bounds for sufficiently large iterations on an increasing function starting from *any* initial point of its domain.

In this paper, we extend the result of Olszewski (2021b) to monotone upper order hemicontinuous correspondences. This is an important extension as mappings studied in many economic settings are typically not single-valued. We consider two important domains for our correspondences: complete lattices and σ-complete lattices.[4] We construct fixed-point lower and upper bounds for the sequences of iterations of a weakly monotone (resp., strongly monotone) upper order hemicon-

[1] See also Knaster and Tarski (1928).

[2] In the Veinott-Zhou theorem, monotone means ascending in the strong set order sense. In Section 2 of this paper, we refer to such correspondences as weakly monotone.

[3] For example, see the results on constructive versions of Tarski's theorem in Cousot and Cousot (1979) and Echenique (2005). To the best of our knowledge, this paper is the first paper in the literature that provides constructive, iterative methods of finding fixed points from the Veinott-Zhou result.

[4] In economic applications, the difference between complete lattices and σ-complete lattices can be important. One such example is a fixed-point problem in spaces of (Borel) measurable functions over a compact domain $A \subset \mathbb{R}^n$, where the space has least and greatest elements, and is endowed with a pointwise partial ordering. This space is generally only σ-complete. When this space is given an almost everywhere pointwise partial ordering, its equivalence classes become a complete lattice (e.g., see Van Zandt (2010), Lemma 5).

tinuous correspondence $F : A \rightrightarrows A$ that transforms a complete lattice (resp., a σ-complete lattice) *starting from any given initial point* $a^0 \in A$. More precisely, we construct fixed points $\underline{a}^*$ and $\overline{a}^*$ such that sufficiently remote elements a^k of any sequence of iterations $(a^k)_{k=0}^{\infty}$ (i.e., such that $a^{k+1} \in F(a^k)$ for all k) are contained between $\underline{a}^*$ and $\overline{a}^*$.[5] As in Olszewski (2021b), the fixed points $\underline{a}^*$ and $\overline{a}^*$ are sharp or tight, i.e., $\underline{b} \leq \underline{a}^*$ and $\overline{a}^* \leq \overline{b}$ if fixed points $\underline{b}$ and $\overline{b}$ are such that remote *finite* iterations of F starting at a^0 are located between $\underline{b}$ and $\overline{b}$.[6]

One might argue that extensions of fixed-point theorems from functions to correspondences have worked pretty well in a variety of settings. In particular, in our setting one may consider iterations $a^{k+1} = \inf F(a^k)$ to obtain the lower fixed-point bound, and iterations $a^{k+1} = \sup F(a^k)$ to obtain the upper fixed-point bound. This is in fact the main idea behind the Veinott-Zhou extension of Tarski's theorem. We show, however, that this idea does *not* deliver the desired extension. More specifically, it would deliver fixed-point bounds, but they would not necessarily be *tight*.

This paper is related to an important and large literature in economics that applies the Tarski-Kantorovich fixed-point theorem when studying the existence of equilibria.[7] The Tarski-Kantorovich theorem says for an order continuous transformation of a countably chain complete partially ordered set (CCPO) with least (resp., greatest) elements, the supremum (resp., infimum) of iterations from the

[5] See his Proposition 1 for a precise assertion.

[6] It is important to note that little is known in the existing literature even about the existence of fixed points for monotone upper order hemicontinuous correspondences that transform sigma-complete lattices. Our arguments here verify the existence, as well as provide iterative tight fixed-point bounds for any initial $a^0 \in A$.

[7] Some examples of work in economics applying the Tarski-Kantorovich theorem include papers on supermodular games (e.g., Van Zandt (2010), Kunimoto and Yamashita (2020), Balbus et al. (2022)), rationalizability in games (e.g. Ok (2004)), models of production chains (Kikuchi et al., 2018), dynamic programming with unbounded returns (e.g, Kamihigashi (2014), Becker and Rincón-Zapatero (2021) among many others), the existence of recursive equilibrium in dynamic stochastic growth models (e.g., Coleman (1991), Mirman et al. (2008), Datta et al. (2018)), computing Bewley models in macroeconomics (e.g., Li and Stachurski (2014), Açıkgöz (2018)).

least (resp., greatest) element of the CCPO will converge in order to the least (resp., greatest) fixed point.[8] One way of understanding the results in Olszewski (2021b) is that he shows that for order continuous functions that transform σ-complete lattice, there exists a generalization of the Tarski-Kantorovich theorem, where from any element of the function's domain, elements of the fixed-point set form tight bounds on sufficiently remote iterations.

This paper extends this idea to the case of correspondences. In particular, we show the Tarski-Kantorovich theorem holds in both: the setting of the original Veinott-Zhou extension of Tarski's theorem for complete lattices, where the correspondence is additionally assumed to be upper order hemicontinuous, as well as in the setting of strongly monotone upper order hemicontinuous correspondences in σ-complete lattices, where the correspondence is additionally required to have a least and a greatest element.

To obtain our results, we introduce a new notion of order continuity for monotone correspondences (i.e., "upper order hemicontinuity"). In Section 5 of this paper, we show the result for the upper order hemicontinuous correspondences can be extended to correspondences with discontinuities but its proof is more involved, requires transfinite constructions, and the extension is perhaps of less interest for economists (particularly in applications). The fact that such transfinite constructions are required without order continuity is not surprising given the literature on constructive characterizations of Tarski's theorem where transfinite arguments appear indispensable (e.g., Cousot and Cousot (1979) or Echenique (2005) among others).

We believe our extensions in this paper are important because most iterations that we consider in economics (and perhaps in other areas of research) use cor-

[8] For example, see Jachymski et al. (2000), Theorem 1, and Dugundji and Granas (1982), p.15 for a discussion of the Tarski-Kantorovich theorem. See also Balbus et al. (2015), Theorem 1.

respondences. For example, players happen to have multiple best responses in games, including those of strategic complementarities, and consumers or producers happen to have multiple optimal bundles. Multiplicity appears when payoffs are not strictly quasi-concave with respect to players' own actions, and consumers' or producers' choices. For example, a small reduction in an oligopolist's price may lower its current profits, but a larger reduction, which lowers the current profits by more, may make other firms exit or deter subsequent entry; or in a contest with multiple prizes whose values are convex, the increase in the expected value of prize induced by a small increase in effort may not be worth the cost of this additional effort, but a larger effort may result in a sufficient increase in prize to compensate for the effort cost.

To illustrate the usefulness of tight fixed-point bound for increasing correspondences we present an application of our results to a class of social learning models on networks (studied recently by Cerreia-Vioglio et al. (2023)). Our tools allow for extending their analysis in the cases in which social learning may not result in converging to a consensus. In particular, our bounds enable us to estimate numerically the limit amount of disagreement. Despite the fact that our bounds use double limits, the iterative numerical procedure that we analyze in our example gave the bounds (up to 5 decimal places) in fewer than 25 iterations.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partial order $\geq$. For $a', a \in A$, we say a' is *strictly higher* than a, and write $a' > a$, whenever $a' \geq a$ and $a' \neq a$. A poset $(A, \geq)$ is a *lattice* if for any $a, a' \in A$ there exist the least upper bound of $\{a, a'\}$ (denoted by $a \vee a'$ or $\sup\{a, a'\}$) and the greatest lower bound of $\{a, a'\}$ (denoted by $a \wedge a'$ or $\inf\{a, a'\}$).

A lattice A is *complete* if there also exist $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$ for all $B \subseteq A$. A lattice A is σ-complete, whenever for any countable $B \subseteq A$, $\bigvee B$ and $\bigwedge B$ exist in A. A subset $B \subseteq A$ is a *sublattice* of A if $a \vee a'$ and $a \wedge a'$, as defined in $(A, \geq)$, belong to B for all $a, a' \in B$. A sublattice B of a lattice A is a *subcomplete sublattice* if for any $C \subseteq B$ the supremum $\bigvee C$ and the infimum $\bigwedge C$, as defined in $(A, \geq)$, belong to B.

We can compare subsets of A using set relations compatible with $(A, \geq)$. Let 2^A denote the set of all subsets of A. If $(A, \geq)$ is a poset, and $B, B' \in 2^A \backslash \{\varnothing\}$, we write $B' \geq^S B$ if for all $b' \in B'$, $b \in B$, $b' \geq b$. If $(A, \geq)$ is a lattice, B and B' are two nonempty subsets of A, we say B' is (Veinott)-*strong set order higher* than B, denoted by $B' \geq^{SSO} B$, whenever for any $b' \in B'$ and $b \in B$, $b' \wedge b \in B$ and $b' \vee b \in B'$.

Let $F : A \rightrightarrows B$ be a nonempty-valued correspondence, where $(A, \geq)$ and $(B, \geq)$ are posets. We say F is *strongly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^S F(a)$. Now, let $(B, \geq)$ be a lattice. We say F is *weakly monotone (increasing)* whenever $a' > a$ implies that $F(a') \geq^{SSO} F(a)$.

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *increasing* if $a^{k+1} \geq a^k$ for all k. It is *strictly increasing* if $a^{k+1} > a^k$ for all k. *Decreasing* and *strictly decreasing* sequences can be defined in the obvious specular manner. A *monotone sequence* then is either increasing or decreasing. We say that an increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ *converges* to $a \in A$ whenever $\bigvee_{k \geq 0} a^k = a$ (resp., $\bigwedge_{k \geq 0} a^k = a$). That is, when a is the supremum (resp., infimum) of the increasing (resp., decreasing) sequence.

We say that a correspondence F is *upper order hemicontinuous* whenever it satisfies the following condition: if a *monotone* sequence $(a^k)_{k=0}^{\infty}$ converges to a, then any *monotone* sequence $(b^k)_{k=0}^{\infty}$ such that $b^k \in F(a^k)$ for all k converges to some $b \in F(a)$. Finally, a function $f : A \to B$ is order-preserving (or increasing) on A if $a \leq a'$ implies $f(a) \leq f(a')$ for any a, a' in A. A function f is *upward order continuous* (resp., *downward order continuous*) if for any increasing (resp., decreasing) convergent sequence $(a^k)_{k=0}^{\infty}$ with $a^k \in A$, we have:

$$f\left(\bigvee_{k\in\mathbb{N}} a^k\right) = \bigvee_{k\in\mathbb{N}} f(a^k) \quad \left(\text{resp. } f\left(\bigwedge_{k\in\mathbb{N}} a^k\right) = \bigwedge_{k\in\mathbb{N}} f(a^k)\right).$$

A function f is *order continuous* if it is both upward and downward order continuous. Notice, if f is upward (resp., downward) order continuous, it is order preserving or increasing.[9]

3 Iterations on monotone upper order hemicontinuous correspondences

In this section, we will state and prove our result under the following two alternative sets of assumptions:

Assumption 1 *A is a complete lattice. $F : A \rightrightarrows A$ is weakly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, $F(a)$ is a subcomplete sublattice of A.*

Assumption 2 *A is a σ-complete lattice. $F : A \rightrightarrows A$ is strongly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, the supremum and the infimum of $F(a)$ belong to $F(a)$.*

Two comments are in order. First, upper order hemicontinuity turns out to be a natural condition that is easy to check in many economic applications. For example, in games of strategic complements (GSCs) where payoff functions are jointly

[9] If a function is upward (resp., downward) order continuous, it is also sup (resp., inf) preserving. So our definitions coincide with standard definitions of order continuity (e.g., Dugundji and Granas (1982), p. 15).

continuous in action profiles, the resulting best-reply correspondences are upper order hemicontinuous as a consequence of well-known maximum theorems (e.g., Berge's theorem). Second, there are settings of interest in which Assumption 2 should be applied. Examples include interim formulation of Bayesian supermodular games, Schmeidler's formulation of large supermodular games or stochastic supermodular games (see e.g. Van Zandt (2010), Balbus et al. (2019) or Balbus et al. (2014)).

For any given $a^0 \in A$, we will first construct a pair of fixed points (denoted by $\underline{a}^*$ and $\overline{a}^*$) of $F : A \rightrightarrows A$. This construction will take a number of steps and lemmas. The two fixed points will be, somewhat imprecisely speaking, tight fixed-point bounds for all iterations of the correspondence F. We will make this assertion precise in the statement of our result.

Define functions $\underline{F} : A \to A$ and $\overline{F} : A \to A$ as follows

$$\underline{F}(a) := \bigwedge F(a) \quad \text{and} \quad \overline{F}(a) := \bigvee F(a).$$

Under Assumption 1, as well as Assumption 2, $\overline{F}$ and $\underline{F}$ are both well-defined selections[10] of F. All lemmas below hold true under Assumption 1 as well as under Assumption 2. We will therefore not explicitly make these assumptions in the statements of the lemmas. The proofs of the lemmas are relegated to Appendix.

Lemma 1 $\overline{F}$ *(resp., $\underline{F}$) is downward order continuous (resp., upward order continuous).*

Let $\underline{a}^1 = \underline{F}(a^0)$ and $\overline{a}^1 = \overline{F}(a^0)$ be the infimum and the supremum of $F(a^0)$; by induction, for $k = 1, 2, \ldots$ let $\underline{a}^{k+1}$ and $\overline{a}^{k+1}$ be the infimum of $F(\underline{a}^k)$ and supremum of $F(\overline{a}^k)$, i.e.

$$\underline{a}^{k+1} = \underline{F}(\underline{a}^k) \quad \text{and} \quad \overline{a}^{k+1} = \overline{F}(\overline{a}^k).$$

[10] A selection of a correspondence $F : A \rightrightarrows B$ is any function $f : A \to B$ such that $f(a) \in F(a)$ for any $a \in A$.

It will be convenient to define $\underline{a}^0$ and $\overline{a}^0$ as a^0. Let $\underline{a}^\omega = \liminf_k \underline{a}^k$ and $\overline{a}^\omega = \limsup_k \overline{a}^k$. That is,

$$\underline{a}^\omega = \lim_k \bigwedge_{l \geq k} \underline{a}^l \quad \text{and} \quad \overline{a}^\omega = \lim_k \bigvee_{l \geq k} \overline{a}^l.$$

Lemma 2 *There exists $a \in F(\underline{a}^\omega)$ such that $a \leq \underline{a}^\omega$ Similarly, there exists $a \in F(\overline{a}^\omega)$ such that $a \geq \overline{a}^\omega$.*

Under Assumption 1, let $\underline{a}^{\omega+1}$ be the supremum of the elements of $F(\underline{a}^\omega)$ that are smaller than $\underline{a}^\omega$, and let $\overline{a}^{\omega+1}$ be the infimum of the elements of $F(\overline{a}^\omega)$ that are greater than $\overline{a}^\omega$; under Assumption 2, let $\underline{a}^{\omega+1}$ be any element of $F(\underline{a}^\omega)$ smaller than $\underline{a}^\omega$, and let $\overline{a}^{\omega+1}$ be any element of $F(\overline{a}^\omega)$ greater than $\overline{a}^\omega$. That is, under Assumption 1,

$$\underline{a}^{\omega+1} = \bigvee F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \quad \text{and} \quad \overline{a}^{\omega+1} = \bigwedge F(\overline{a}^\omega) \cap J(\overline{a}^\omega),$$

with $I(a) := \{a' \in A : a' \leq a\}$ and $J(a) := \{a' \in A : a' \geq a\}$.

By Lemma 2, $F(\underline{a}^\omega) \cap I(\underline{a}^\omega) \neq \emptyset$, and the same is true for $F(\overline{a}^\omega) \cap J(\overline{a}^\omega)$. Hence, by each of our two assumptions, both $\underline{a}^{\omega+1}$ and $\overline{a}^{\omega+1}$ are well defined elements of $F(\underline{a}^\omega)$, respectively of $F(\overline{a}^\omega)$. Note that we apply here that the condition from assumption 1 which says that $F(a)$ is a sublattice.

We can now continue our iterations starting from $\underline{a}^\omega$ and $\overline{a}^\omega$. We define the following sequences $(\underline{a}^{\omega+k})_{k=1}^\infty$ and $(\overline{a}^{\omega+k})_{k=1}^\infty$ recursively as follows:

$$\underline{a}^{\omega+k+1} = \bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \quad \text{and} \quad \overline{a}^{\omega+k+1} = \bigwedge F(\overline{a}^{\omega+k}) \cap J(\overline{a}^{\omega+k}),$$

under Assumption 1; under Assumption 2, $\underline{a}^{\omega+k+1}$ is any element of $F(\underline{a}^{\omega+k})$ such that $\underline{a}^{\omega+k+1} < \underline{a}^{\omega+k}$, and $\overline{a}^{\omega+k+1}$ is any element of $F(\overline{a}^{\omega+k})$ such that $\overline{a}^{\omega+k+1} > \overline{a}^{\omega+k}$, unless $\underline{a}^{\omega+k}$ (resp., $\overline{a}^{\omega+k}$) is a fixed point, in which case $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ (resp., $\overline{a}^{\omega+k+1} = \overline{a}^{\omega+k}$).

This yields the following results:

Lemma 3 *The sequences* $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ *and* $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ *are both well-defined. Moreover, if any* $\underline{a}^{\omega+k_0}$ *(resp.,* $\overline{a}^{\omega+k_0}$*) is a fixed point of* F*, then the sequence* $\left(\underline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$ *(resp.,* $\left(\overline{a}^{\omega+k}\right)_{k=k_0}^{\infty}$*) is constant.*

Lemma 4 *(i) The sequence* $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ *is decreasing, and its limit* $\underline{a}^*$ *is a fixed point of* F*; (ii) the sequence* $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ *is increasing and its limit* $\overline{a}^*$ *is a fixed point of* F*.*

This completes our construction of fixed points $\underline{a}^*$ and $\overline{a}^*$. It possibly appears as a puzzling feature of the construction that $\underline{a}^{k+1}$ is defined as the infimum of $F(\underline{a}^k)$, while $\underline{a}^{\omega+k+1}$ is defined under Assumption 1 as the supremum of $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. (A similar question concerns $\overline{a}^{k+1}$ and $\overline{a}^{\omega+k+1}$.) For the definition of $\underline{a}^{k+1}$ we had no choice. It had to be the infimum of $F(\underline{a}^k)$ to guarantee $\underline{a}^*$ is indeed a lower bound for the large iterations of F. In turn, if we defined $\underline{a}^{\omega+k+1}$ as the infimum of $F(\underline{a}^{\omega+k})$, then $\underline{a}^*$ would still be a fixed-point lower bound for the iterations of F, but it could not be the sharp one. This is illustrated by the following example.

Example 1 *Recall Example 1 from Olszewski (2021b) in which* X *is a sublattice of* $\mathbb{R}^2$ *equipped with the coordinate-by-coordinate ordering that consists of points: (-1,0), (0,0), (1,0), (0,1), (1,1), (2,1). Olszewski defined a function* $f : X \to X$ *such that* $\underline{a}^{\omega} = \liminf f^k(0,1) = (0,0)$*, but* $f(0,0) = (-1,0)$*.*

Consider a sublattice $A = X \cup I$ *of* $\mathbb{R}^2$*, where* $I = \{(y,0) : y \in [-4,-1]\}$*, equipped with the coordinate-by-coordinate ordering. Extend function* $f \mid X - \{(-1,0)\}$ *to a correspondence* $F : A \rightrightarrows A$ *by letting* $F(y,0) = \{(z,0) : z \in [-4,-2]\}$ *for* $y \in [-3,-1]$ *and* $F(y,0) = (-4,0)$*for* $y \in [-4,-3)$*.*[11] *That is,* $F = f$ *on* $X - I$*, and* F *on* I *is illustrated in Figure 1, in which we identified* I *with the interval* $[-4,-1]$*.*

[11] Note that $F(-1,0) = \{(z,0) : z \in [-4,-2]\}$, while $f(-1,0) = (-1,0)$ in Olszewski (2021b).

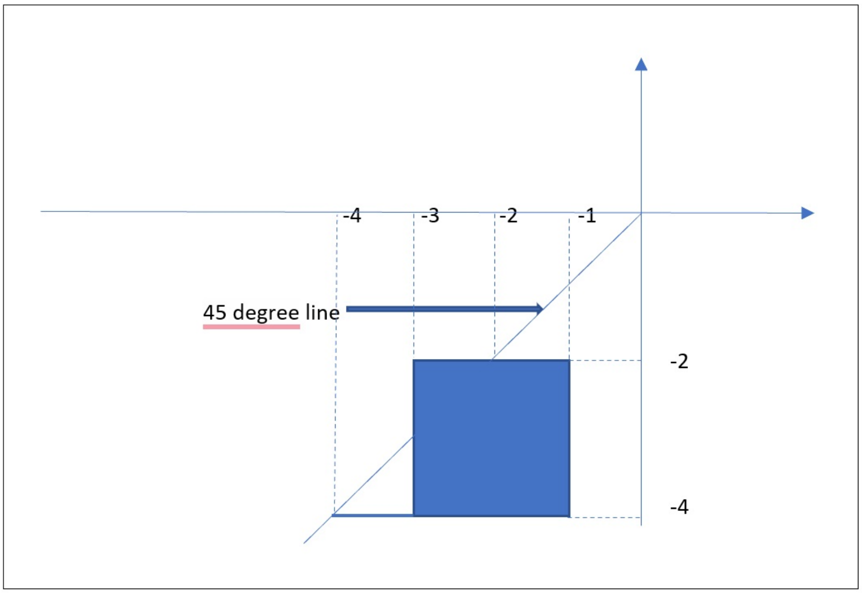

Figure 1: The graph correspondence $F \mid I$ from Example 1.

If we defined $\underline{a}^{\omega+1}$ *as* $\inf F(\underline{a}^{\omega}) = (-4, 0)$*, then we would obtain* $\underline{a}^* = (-4, 0)$*, and this would not be a sharp fixed-point bound for the sequence* $(a^k)_{k=0}^{\infty}$*. This sharp fixed-point bound is* $\underline{a}^* = (-2, 0)$*, and this* $\underline{a}^*$ *is indeed obtained if* $\underline{a}^{\omega+1}$ *is defined as* $\bigvee F(\underline{a}^{\omega}) \cap I(\underline{a}^{\omega}) = (-2, 0)$*, as we do.*

Under Assumption 2, $\underline{a}^{\omega+k+1}$ and $\overline{a}^{\omega+k+1}$ can be defined as the infimum of $F(\underline{a}^{\omega+k})$, because for strongly monotone F, if $\underline{a}^{\omega+k}$ is not a fixed point of F, then $\sup F(a) \leq \inf F(\underline{a}^{\omega+k})$ for all $a \in F(\underline{a}^{\omega+k})$. So, no element of $F(\underline{a}^{\omega+k})$ can be a fixed point possibly except $\inf F(\underline{a}^{\omega+k})$. Thus, by defining $\underline{a}^{\omega+k+1}$ in the way in which we do, we can be sure that we will not "jump down" over any fixed point.

We can now state and prove the following key result.

Proposition 1 *Both under Assumption 1 and under Assumption 2, the following statements hold true: (i) An increasing sequence* $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ *converges to* $\underline{a}^{\omega} \geq \underline{a}^*$*,* *and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$ *for all* k*, we have that* $\bigwedge_{l\geq k} \underline{a}^l \leq a^k$*. A decreasing sequence* $\left(\bigvee_{l\geq k} \overline{a}^l\right)_{k=0}^{\infty}$ *converges to* $\overline{a}^\omega \leq \overline{a}^*$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$ *for all* k*, we have that* $a^k \leq \bigvee_{l\geq k} \overline{a}^l$*.*

(ii) Suppose that $\underline{b}$ *is fixed point of* F *for which there exist an increasing sequence* $(\underline{b}^k)_{k=1}^{\infty}$ *such that* $\lim_k \underline{b}^k \geq \underline{b}$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$*, we have that* $\underline{b}^k \leq a^k$ *for all* k*, then* $\underline{b} \leq \underline{a}^*$*. Suppose that* $\overline{b}$ *is fixed point of* F *for which there exist an decreasing sequence* $(\overline{b}^k)_{k=1}^{\infty}$ *such that* $\lim_k \overline{b}^k \leq \overline{b}$*, and for any sequence* $(a^k)_{k=0}^{\infty}$ *such that* $a^{k+1} \in F(a^k)$*, we have that* $a^k \leq \overline{b}^k$ *for all* k*, then* $\overline{b} \geq \overline{a}^*$*.*

Proof: We will prove the theorem for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. Part (i) follows directly from the definitions and previous results. We will prove part (ii). Since $\underline{b}^l \leq a^l$ for all l, $\bigwedge_{l\geq k} \underline{b}^l \leq \bigwedge_{l\geq k} \underline{a}^l$; and since the sequence $(\underline{b}^k)_{k=1}^{\infty}$ is increasing, $\underline{b}^k = \bigwedge_{l\geq k} \underline{b}^l$, therefore $\lim_k \underline{b}^k \leq \lim_k \bigwedge_{l\geq k} \underline{a}^l = \underline{a}^\omega$. Thus, $\underline{b} \leq \underline{a}^\omega$. This completes the proof if $\underline{a}^* = \underline{a}^\omega$. If not, then $\underline{a}^\omega$ is not a fixed point and $\underline{b} < \underline{a}^\omega$. Recall $\underline{b} \in F(\underline{b})$, and $\underline{a}^{\omega+1} \in F(\underline{a}^\omega)$. Under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^\omega)$ because $F(\underline{b}) \leq^{SSO} F(\underline{a}^\omega)$. Since $\underline{b} < \underline{a}^\omega$, and by Lemma 4, $\underline{a}^{\omega+1} < \underline{a}^\omega$, we have that $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^\omega$. This implies that $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^\omega) \cap I(\underline{a}^\omega)$. Since $\underline{a}^{\omega+1}$ is the greatest element of this set, hence $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega+1}$. So $\underline{b} \leq \underline{a}^{\omega+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+1}$ because $\underline{b} < \underline{a}^\omega$, so the strong monotonicity of F implies $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^\omega) \leq \underline{a}^{\omega+1}$. We show $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and consequently $\underline{b} \leq \underline{a}^*$. We have proven this for $k = 1$; suppose it is the case for some k. The proof is complete if $\underline{a}^{\omega+k}$ is a fixed point, because by Lemma 3, $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$. If $\underline{a}^{\omega+k}$ is not a fixed point, $\underline{b} < \underline{a}^{\omega+k}$, and $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$. Moreover, $\underline{b} \vee \underline{a}^{\omega+k+1} \in I(\underline{a}^{\omega+k})$, hence $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k+1}$ was defined as the greatest element of this set under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+k+1} \leq \underline{a}^{\omega+k+1}$, consequently $\underline{b} \leq \underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+k+1}$ because $\underline{b} < \underline{a}^{\omega+k}$, so strong monotonicity of F implies that $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^{\omega+k}) \leq \underline{a}^{\omega+k+1}$. Thus, $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and also $\underline{b} \leq \underline{a}^*$. ∎

Proposition 1 captures formally the intuition that $\underline{a}^*$ and $\overline{a}^*$ are tight fixed-point bounds between which sufficiently large iterations of F are located.

Remark. In Proposition 1, we could alternatively require the sequence $(\underline{b}^k)_{k=1}^{\infty}$ to be decreasing, and the sequence $(\overline{b}^k)_{k=1}^{\infty}$ to be increasing. (Recall that we define no other convergent sequences.) Then, $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ would be such a decreasing sequence for $\underline{a}^*$, and $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ would be such an increasing sequence for $\overline{a}^*$. The hypothesis of Proposition 1 would still hold true, because $\underline{b}^k \leq \underline{a}^k$ for all k implies that

$$\underline{b} \leq \lim_k \underline{b}^k \leq \liminf_k \underline{a}^k = \underline{a}^{\omega}.$$

Then, the arguments analogous to those from the proof of Proposition 1 yield $\underline{b} \leq \underline{a}^{\omega+k}$ for all k, which implies that $\underline{b} \leq \underline{a}^*$. The proof that $\overline{b} \geq \overline{a}^*$ is analogous.

Remark. The Tarski-Kantorovich theorem says that if A is a CCPO with a least element $\underline{a}$ and a greatest element $\overline{a}$, and $f : A \to A$ is an order continuous function, then f has a least fixed point: $\underline{a}^* = \sup_k\{f^k(\underline{a})\}$ and a greatest fixed point: $\overline{a}^* = \inf_k\{f^k(\overline{a})\}$. A natural way to generalize this principle to a correspondence $F : A \rightrightarrows A$ is to consider some iterations of its least selection: $\underline{F}(a) = \inf F(a)$ and its greatest selection $\overline{F}(a) = \sup F(a)$ from least and greatest elements of A. If the least selection exists and is upward order continuous, then F has a least fixed point $\underline{a}^* = \sup_k\{\underline{F}^k(\underline{a})\}$. Similarly, if the greatest selection exists and is downward order continuous, then F has a greatest fixed point $\overline{a}^* = \inf_k\{\overline{F}^k(\overline{a})\}$. Our Proposition 1 implies that under Assumption 1 or 2 both these fixed points exist.

4 Application: Social learning on networks

DeGroot's model, in which agents take weighted averages of the opinions they observe, is a commonly applied approach to studying social learning on networks. Obviously, this very specific type of learning cannot well describe all real-life situations of interest. Cerreia-Vioglio et al. (2023) recently suggested a more general model, in which an opinion aggregator is a function that satisfies certain axioms. In their model of an economy of n agents, an opinion profile is represented by a vector $a \in [0,1]^n$, and learning is represented by an opinion aggregator $T : [0,1]^n \to [0,1]^n$ that is monotone[12] with respect to coordinate-by-coordinate ordering on $[0,1]^n$.

We now illustrate the usefulness of our results by applying them to the setting studied by Cerreia-Vioglio et al. (2023). A group of agents $N = \{1,2,3\}$ share their opinions $a^0 \in [0,1]^3$. The weights assigned to the other agents are represented by the matrix:

$$W = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}.$$

That is, the entries in row i of the matrix represent the weights assigned by agent i to the opinions of all agents. The average aggregator $T^1(a)$ is defined, agent by agent, as the sum of opinions multiplied by their weights. For example, agent 1 assigns weight 0.4 to her own opinion and weight 0.3 to the opinion of each other agent. So, if $a^0 = (0.8, 0.6, 0.4)$, then

$$a^1 = T^1(a^0) = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.62 \\ 0.5 \\ 0.56 \end{bmatrix}.$$

[12] In addition to monotonicity, they impose two other axioms: normalization ($T(k,\dots,k) = (k,\dots,k)$ for all $k \in [0,1]$) and translation invariance ($T(x_1+k,\dots,x_n+k) = T(x_1,\dots,x_n) + (k,\dots,k)$ whenever it makes sense). They all are satisfied in our application.

The aggregation through weighted averages satisfies the conditions required by Cerreia-Vioglio et al. (2023). It also satisfies Assumptions 1 and 2 of our paper. It always achieves limit consensus. For example, if $a^0 = (0.8, 0.6, 0.4)$, then it can be computed numerically that the long run opinion $\lim_{n\to\infty} T^1_n(a^0)$ of all agents is 0.54286.

Consider now the aggregation induced by the median. This operator is defined as follows: the opinion of each agent in a vector of opinions a is assigned a probability equal to the agent's weight. This determines a probability distribution over opinions. $T^2(a)$ is defined, agent by agent, as the median of this distribution. For example, $a^1 = T^2(a^0) = (0.6, 0.4, 0.6)$ for $a^0 = (0.8, 0.6, 0.4)$. The median aggregator also satisfies the conditions required by of Cerreia-Vioglio et al. (2023). Actually, both weighted averages and medians were used as examples in their paper. Except the non-generic matrices, such that the sum of a proper subset of entries in some row is equal to 0.5, the median is always unique and the median aggregator is continuous and satisfies our Assumptions 1 and 2. For the remaining non-generic matrices, the median aggregator is upper order hemicontinuous and satisfies our Assumption 1.[13]

When $a^0 = (0.8, 0.6, 0.4)$, $a^k = (0.6, 0.4, 0.6)$ for odd k and $a^k = (0.6, 0.6, 0.4)$ for even k. Thus,

$$\liminf_{k=\infty} a^k = (0.6, 0.4, 0.4) \text{ and } \limsup_{k=\infty} a^k = (0.6, 0.6, 0.6).$$

Despite the fact that no limit consensus is reached, by looking at lim inf and lim sup we can say that the limit disagreement is only of size 0.2, that is, it is only a half of the initial disagreement.

Note that while $\limsup_{k=\infty} a^k$ is a fixed point of the aggregator T^2 (namely $\overline{a}^*$) $\liminf_{k=\infty} a^k$ is not a fixed point; indeed, $T^2(0.6, 0.4, 0.4) = (0.4, 0.4, 0.4)$.

[13] For all a, $T^2(a)$ is a product of intervals.

Therefore applying the second rounds of iterations of T^2 to $\liminf$ is necessary (not necessary for $\limsup$) if the objective is to find the fixed-point bound (namely $\underline{a}^*$). The fixed-point bound will be useful when a modeler believes that a stable outcome should be reached and allows for a broader class of adaptive dynamics.[14]

Up to now most of the aggregators considered in this application were functions. However, in some settings correspondence aggregators seem more appropriate. Suppose, for example, that a modeler only wants to assume that agents somehow average opinions in the process of adaptive learning. She may then use T that takes values between the weighted average and the median. More precisely, let

$$T(a) = \{y : T^1(a) \wedge T^2(a) \leq y \leq T^1(a) \vee T^2(a)\};$$

for example,

$$T(0.8, 0.6, 0.4) = [0.6, 0.62] \times [0.4, 0.5] \times [0.56, 0.6].$$

This correspondence T is weakly monotone, because so are T^1 and T^2. The values of T are products of intervals. And T is upper order hemicontinuous, because so are T^1 and T^2. So our Assumption 1 is satisfied.[15] Finally, one can estimate the size of the limit disagreement by computing numerically that

$$\underline{a}^* = \lim\inf_{k=\infty} \underline{a}^k = (0.4, 0.4, 0.4) \text{ and } \lim\sup_{k=\infty} \overline{a}^k = (0.6, 0.6, 0.6) = \overline{a}^*.$$

The above examples suggests that the two tight fixed-point bounds represent the social consensus. The following example shows it is not always the case. To

[14] An aggregator that takes as input only the most recent opinions generate an extreme form of adaptive learning. In some settings, it makes sense to apply an aggregator T to the minimum of opinions of each agent in two (or more) previous periods. This results, for the median aggregator, in the convergence to the consensus $(0.4, 0.4, 0.4)$. We postpone more detailed analysis of other forms of adaptive learning for future research. However, it seems that our fixed points are the right bounds for richer classes of adaptive learning processes.

[15] Note also that $T(k, \ldots, k) = \{(k, \ldots, k)\}$ for all $k \in [0, 1]$. So, normalization is satisfied, and so is the version of translation invariance for correspondences.

see that, let:

$$W = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.2 & 0.7 & 0.1 \end{bmatrix},$$

and let the initial profile be $a^0 = (0.8, 0.4, 0.6)$. Suppose now that each agent ignores the extremal 20% of the opinions and so updates its opinion somewhere in the interval between 0.2 and 0.8 percentile of the opinion distribution.

Computing the upper iterations, i.e. iterating on the 0.8 percentile aggregator, we immediately obtain that $\overline{a}^t = (0.8, 0.6, 0.6)$ for any $t \geq 1$ and, although no consensus is reached, a profile $(0.8, 0.6, 0.6)$ is the upper fixed-point bound of our learning process.

For the lower iterations, we obtain: $\underline{a}^t = (0.6, 0.6, 0.4)$ for odd t and $\underline{a}^t = (0.6, 0.4, 0.6)$ for even t. The learning cycles, but

$$\lim \inf_{k=\infty} \underline{a}^k = (0.6, 0.4, 0.4).$$

Now:

$$T(\lim \inf_{k=\infty} \underline{a}^k) = [0.4, 0.6] \times [0.4, 0.6] \times [0.4, 0.6].$$

Interestingly, both: the least element of $T(\lim\inf_{k=\infty} \underline{a}^k)$, namely $(0.4, 0.4, 0.4)$, as well as its greatest element, namely $(0.6, 0.4, 0.4)$, are fixed points of T. But the tight lower fixed-point bound is $(0.6, 0.4, 0.4)$. That is in line with our construction on pages 9-10. The disagreement between both sharp fixed-point bounds is hence 0.2.

5 Iterations on discontinuous correspondences

There are two possible extensions of Proposition 1. First, one may ask if there exist tight fixed-point bound for sequences of *finite* iterations starting from an arbitrary point of a lattice. The answer to this question is negative, even for functions, as the following example shows.

Example 2 *Let* $A = [0,1) \cup \{2 - 1/n : n = 1, 2, \ldots\} \cup \{2, 3\}$ *with the lattice structure inherited from the reals. Let* $f : A \to A$ *be given by* $f(a) = a$ *for* a *from* $[0, 1)$, $f(a) = 2 - 1/(n + 1)$ *for* $a = 2 - 1/n$, *and* $f(a) = 3$ *for* $a = 2, 3$. *Points* $a < 1$ *and* $a = 3$ *are the fixed points of function* f. *For* $a^0 = 1$, *the sequence of finite iterations* $a^n = f^n(a^0) = 2 - 1/(n+1)$ *is increasing and converges to* $a = 2$. *Thus,* $a = 3$ *is the tight fixed-point upper bound for this sequence of iterations, and any* $a < 1$ *is a fixed-point lower bound. This implies that the tight fixed-point lower bound does not exist.*

Secondly, one may wonder whether our results can be extended to discontinuous weakly monotone correspondences. In fact Proposition 1 can be extended in this way, but the cost of relaxing our continuity condition is that we must introduce transfinite iterations. In addition, we must restrict attention to iterating correspondences that transform complete (not all sigma-complete) lattice A. More precisely, the following result can be obtained by minimally modifying the proof from Olszewski (2021a).

Let $\alpha > |A|$, where $|A|$ stands for the cardinality of A, be a cardinal number. For every $\underline{a}_0 = \overline{a}^0 = a^0 \in A$, and every weakly monotone correspondence $F : A \rightrightarrows A$, say that $(a_\beta)_{\beta<\alpha}$ is *a sequence of transfinite iterations* of F if:

$$a_\beta \in F(a_{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta - 1;$$

and

$$\bigvee_{\gamma<\beta} \bigwedge_{\gamma\leq\delta<\beta} a_\delta \leq a_\beta \leq \bigwedge_{\gamma<\beta} \bigvee_{\gamma\leq\delta<\beta} a^\delta \text{ if } \beta \text{ is a limit ordinal.}$$

In addition, distinguish two special sequences of transfinite iterations

$$\underline{a}^{\beta} =: \begin{cases} \inf F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta - 1 \\ \bigwedge\limits_{\gamma<\beta} \bigvee\limits_{\gamma\leq\delta<\beta} a^{\delta} \text{ if } \beta \text{ is a limit ordinal.} \end{cases} \tag{1}$$

and

$$\overline{a}^{\beta} =: \begin{cases} \sup F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta - 1 \\ \bigwedge\limits_{\gamma<\beta} \bigvee\limits_{\gamma\leq\delta<\beta} a^{\delta} \text{ if } \beta \text{ is a limit ordinal.} \end{cases} \tag{2}$$

Proposition 2 *Suppose that* $(A, \leq)$ *is a complete lattice, and* $F : A \rightrightarrows A$ *is a weakly monotone correspondence such that* $F(a)$ *has the smallest and the greatest element for all* $a \in A$*. Let* $\alpha > |A|$ *be a regular cardinal number.*[16] *Then, for any* $a_0 = a^0 \in A$*, there exist* $\underline{\beta}, \overline{\beta} < \alpha$ *such that* $\underline{a}_{\beta} = \underline{a}_{\underline{\beta}}$ *for all* $\underline{\beta} \leq \beta < \alpha$*, and* $\overline{a}^{\beta} = \overline{a}^{\overline{\beta}}$ *for all* $\overline{\beta} \leq \beta < \alpha$*. In particular,* $\underline{a}_{\underline{\beta}}$ *and* $\overline{a}^{\overline{\beta}}$ *are fixed points of* F*.*

Moreover, $\underline{a}_{\underline{\beta}}$ *is the greatest fixed point* $\underline{a}$ *of* F *with the property that* $\underline{a} \leq a_{\beta}$ *for sufficiently large* $\beta < \alpha$ *and for all sequences of transfinite iterations* $(a_{\beta})_{\beta<\alpha}$*, and* $\overline{a}^{\overline{\beta}}$ *and the smallest fixed point* $\overline{a}$ *of* F *with the property that* $a^{\beta} \leq \overline{a}$ *for sufficiently large* $\beta < \alpha$ *and for all sequences of transfinite iterations* $(a_{\beta})_{\beta<\alpha}$*.*

It is possible to obtain a somewhat stronger result than Proposition 2, which requires a somewhat more involved proof. However, since transfinite sequences are unlikely to be of interest for economists, we will not present and discuss this result in this paper.

[16] A regular cardinal number α is defined by the following property: No set of cardinality α can be represented as the union of a family of subsets such that each subset from the family has a cardinality smaller than α, and the family itself is of a cardinality smaller than α.

6 Appendix

Proof of Lemma 1. Since F is weakly increasing, so are $\overline{F}$ and $\underline{F}$. Indeed, if $a' < a''$ then $\underline{F}(a') \wedge \underline{F}(a'') \in F(a')$. As a result,

$$\underline{F}(a') \leq \underline{F}(a') \wedge \underline{F}(a'').$$

Hence $\underline{F}(a') = \underline{F}(a') \wedge \underline{F}(a'') \leq \underline{F}(a'')$. Similarly, we show that $\overline{F}$ is increasing.

We prove the upward continuity of $\underline{F}$. The proof is the same under Assumption 1 and under Assumption 2. Let $(a^k)_{k=1}^{\infty}$ be an increasing sequence in A such that $a = \bigvee_{k\in\mathbb{N}} a^k$. Let $b^k := \underline{F}(a^k)$. Then $b^k \in F(a^k)$ for all $k \in \mathbb{N}$, and $(b^k)_{k=1}^{\infty}$ is increasing. Let $b := \bigvee b^k$. Since b^k belongs to $F(a^k)$ and the sequence $(b^k)_{k=1}^{\infty}$ is increasing, b belongs to $F(a)$ by upper hemicontinuity of F. Hence, $\underline{F}\,(a) \leq b$. On the other hand, $\underline{F}(a) = \underline{F}(\bigvee_{k\in\mathbb{N}} a^k) \geq b^k$ for any k because $\underline{F}$ is increasing. Hence $b \leq \underline{F}(a)$. Together with $\underline{F}\,(a) \leq b$, we have $b = \underline{F}(a)$, and hence the upward continuity. We omit a similar proof that $\overline{F}$ is downward continuous. ■

Proof of Lemma 2. We will prove the lemma for $\underline{a}^{\omega}$; the proof for $\overline{a}^{\omega}$ is analogous. The sequence $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ is increasing, and $\underline{a}^{\omega}$ is its supremum. Let $b^k = \underline{F}\left(\bigwedge_{l\geq k} \underline{a}^l\right)$. By Lemma 1, $\underline{F}$ is an increasing, upward continuous function, hence $(b^k)_{k=1}^{\infty}$ is increasing as well. In addition,

$$a := \bigvee_{k\in\mathbb{N}} b^k = \underline{F}(\underline{a}^{\omega}) \in F(\underline{a}^{\omega}).$$

To finish the proof, we must show that $a \leq \underline{a}^{\omega}$. Since $\bigwedge_{l\geq k} \underline{a}^l \leq \underline{a}^l$ for all $l \geq k$, we have that $b^k \leq \underline{a}^{l+1}$ for all $l \geq k$ by the monotonicity of $\underline{F}$ and the definition of $\underline{a}^{l+1}$ and b^k. So, $b^k \leq \bigwedge_{l\geq k+1} \underline{a}^l \leq \underline{a}^{\omega}$, which gives that $a = \lim_k b^k \leq \underline{a}^{\omega}$. ■

Proof of Lemma 3. We will show the hypothesis for the sequence $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$; the proof for the sequence $\left(\overline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is analogous. That is, we will show by induction that $\underline{a}^{\omega+k+1}$ is well-defined for any $k \geq 0$, and if $\underline{a}^{\omega+k}$ is a fixed point, then $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$.

For $k = 0$, this holds true by Lemma 2. First, suppose that $\underline{a}^{\omega+k}$ is a fixed point of F for some $k > 0$. Then $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$, so $\underline{a}^{\omega+k+1}$ is well-defined by Assumption 1. In addition, $\underline{a}^{\omega+k}$ must be $\bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Hence $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ by the definition of $\underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as $\underline{a}^{\omega+k}$.

Suppose now that $\underline{a}^{\omega+k}$ is not a fixed point of F. By induction hypothesis $\underline{a}^{\omega+k-1}$ is neither a fixed point of F, because then $\underline{a}^{\omega+k} = \underline{a}^{\omega+k-1}$ would also be a fixed point. Hence $\underline{a}^{\omega+k-1} > \underline{a}^{\omega+k}$. By Assumption 1, $F(\underline{a}^{\omega+k}) \leq^{SSO} F(\underline{a}^{\omega+k-1})$. Take any $a' \in F(\underline{a}^{\omega+k})$. Such an a' exists because F is non-empty valued. Since $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k-1})$, it must be that $a' \wedge \underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k})$ and obviously $a' \wedge \underline{a}^{\omega+k} \in I(\underline{a}^{\omega+k})$. As a result $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$. Thus, $\underline{a}^{\omega+k+1}$ is well-defined. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as an arbitrary element of $F(\underline{a}^{\omega+k})$ smaller than $\underline{a}^{\omega+k}$. Such an element exists because F is non-empty valued and strongly monotone. So, $\underline{a}^{\omega+k+1}$ is well-defined. ■

Proof of Lemma 4. We will prove this lemma for $\underline{a}^*$; the proof for $\overline{a}^*$ is analogous. By construction, $\left(\underline{a}^{\omega+k}\right)_{k=0}^{\infty}$ is a decreasing sequence. Let $\underline{a}^*$ be its limit. Since $\underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$ for all k, by taking a limit as $k \to \infty$ and applying the upper hemicontinuity of F we obtain $\underline{a}^* \in F(\underline{a}^*)$.

References

Açıkgöz, O. T. (2018): "On the existence and uniqueness of stationary equilibrium in Bewley economies with production," *Journal of Economic Theory*, 173, 18–55.

Balbus, Ł., P. Dziewulski, K. Reffet, and Ł. Woźny (2019): "A qualitative theory of large games with strategic complementarities," *Economic Theory*, 67, 497–523.

——— (2022): "Markov distributional equilibrium dynamics in games with complementarities and no aggregate risk," *Theoretical Economics*, 17, 725–762.

Balbus, Ł., K. Reffett, and Ł. Woźny (2014): "A constructive study of Markov equilibria in stochastic games with strategic complementarities," *Journal of Economic Theory*, 150, 815–840.

——— (2015): "Time consistent Markov policies in dynamic economies with quasi-hyperbolic consumers," *International Journal of Game Theory*, 44, 83–112.

Becker, R. A. and J. P. Rincón-Zapatero (2021): "Thompson aggregators, Scott continuous Koopmans operators, and least fixed point theory," *Mathematical Social Sciences*, 112, 84–97.

Cerreia-Vioglio, S., R. Corrao, and G. Lanzani (2023): "Dynamic opinion aggregation: long-run stability and disagreement," *The Review of Economic Studies*, forthcoming.

Coleman, W. (1991): "Equilibrium in a production economy with an income tax," *Econometrica*, 59, 1091–1104.

Cousot, P. and R. Cousot (1979): "Constructive versions of Tarski's fixed point theorems," *Pacific Journal of Mathematics*, 82, 43–57.

Datta, M., K. Reffett, and Ł. Woźny (2018): "Comparing recursive equilibrium in economies with dynamic complementarities and indeterminacy," *Economic Theory*, 66, 593–626.

Dugundji, J. and A. Granas (1982): *Fixed Point Theory*, Polish Scientific Publishers.

Echenique, F. (2005): "A short and constructive proof of Tarski's fixed-point theorem," *International Journal of Game Theory*, 33, 215–218.

Jachymski, J., L. Gajek, and P. Pokarowski (2000): "The Tarski–Kantorovitch prinicple and the theory of iterated function systems," *Bulletin of the Australian Mathematical Society*, 20, 247–261.

Kamihigashi, T. (2014): "Elementary results on solutions to the bellman equation of dynamic programming: existence, uniqueness, and convergence," *Economic Theory*, 56, 251–273.

Kikuchi, T., K. Nishimura, and J. Stachurski (2018): "Span of control, transaction costs, and the structure of production chains," *Theoretical Economics*, 13, 729–760.

Knaster, B. and A. Tarski (1928): "Un théoremè sur les fonctions d'ensembles," *Annales de la Societe Polonaise Mathematique*, 6, 133–134.

Kunimoto, T. and T. Yamashita (2020): "Order on types based on monotone comparative statics," *Journal of Economic Theory*, 189, 105082.

Li, H. and J. Stachurski (2014): “Solving the income fluctuation problem with unbounded rewards,” *Journal of Economic Dynamics and Control*, 45, 353–365.

Mirman, L., O. Morand, and K. Reffett (2008): “A qualitative approach to Markovian equilibrium in infinite horizon economies with capital,” *Journal of Economic Theory*, 139, 75–98.

Ok, E. A. (2004): “Fixed set theory for closed correspondences with applications to self-similarity and games,” *Nonlinear Analysis: Theory, Methods & Applications*, 56, 309–330.

Olszewski, W. (2021a): “On convergence of sequences in complete lattices,” *Order*, 38, 251–255.

——— (2021b): “On sequences of iterations of increasing and continuous mappings on complete lattices,” *Games and Economic Behavior*, 126, 453–459.

Tarski, A. (1955): “A lattice-theoretical fixpoint theorem and its applications,” *Pacific Journal of Mathematics*, 5, 285–309.

Van Zandt, T. (2010): “Interim Bayesian Nash equilibrium on universal type spaces for supermodular games,” *Journal of Economic Theory*, 145, 249–263.

Veinott (1992): *Lattice programming: qualitative optimization and equilibria*, Technical Report, Stanford.

Zhou, L. (1994): “The set of Nash equilibria of a supermodular game is a complete lattice,” *Games and Economic Behavior*, 7, 295–300.