A Tarski-Kantorovich Theorem for Correspondences^{*}

Łukasz Balbus[†]

Wojciech Olszewski[‡]

Kevin Reffett[§]

Łukasz Woźny[¶]

December, 2024

Abstract

For a weakly monotone (resp., strongly monotone) upper order hemicontinuous correspondence $F : A \rightrightarrows A$, where A is a complete lattice (resp., a σ -complete lattice), we provide tight fixed-point bounds for sufficiently large iterations $F^k(a^0)$, starting from any point $a^0 \in A$. Our results, hence, prove a generalization of the Tarski-Kantorovich theorem. We provide an application of our results to a class of social learning models on networks.

Keywords: iterations of monotone correspondences; Tarski's fixed-point theorem; Veinott-Zhou version of Tarski's theorem for correspondences; Tarski-Kantorovich theorem for correspondences; adaptive learning. **JEL classification:** C62, C65, C72

^{*} We would like to thank John Quah for his comments on the earlier draft of this paper. Special thanks to Ben Golub for suggesting the application we analyze in Section 4. Lukasz Woźny acknowledges financial support by the National Science Center, Poland: NCN grant number UMO-2019/35/B/HS4/00346. Reffett acknowledges the Dean's Summer Research Award at the WP Carey School of Business at ASU for its financial support.

[†] Faculty of Mathematics, Computer Sciences and Econometrics, University of Zielona Góra, Poland.

 $^{^\}ddagger$ Department of Economics, Northwestern University, USA.

[§] Department of Economics, Arizona State University, USA.

 $[\]P$ Department of Quantitative Economics, SGH Warsaw School of Economics, Poland.

1 Introduction

The celebrated Tarski (1955)¹ fixed-point theorem has found numerous applications in various disciplines, including economics. The theorem states that an increasing transformation of a complete lattice has a complete lattice of fixed points. This result has been extended to the case of weakly monotone correspondences in the work of Veinott (1992) and Zhou (1994a).² In the case of Tarski's original fixed-point theorem, the lowest fixed point is the "limit" of the sequence of iterations starting from the lowest element of the lattice, and the highest fixed point is the "limit" of the sequence of iterations starting from the highest element of the lattice.³ In his two recent papers, Olszewski (2021a,b) characterized the elements of the lattice that are the sharp fixed-point bounds for sufficiently large iterations on an increasing function starting from *any* initial point of its domain.

In this paper, we extend the result of Olszewski (2021b) to monotone upper order hemicontinuous correspondences. This is an important extension as mappings studied in many economic settings are typically not single-valued. We consider two important domains for our correspondences: complete lattices and σ -complete lattices.⁴ We construct fixed-point lower and upper bounds for the sequences of iterations of a weakly monotone (resp., strongly monotone) upper order hemicon-

¹ See also Knaster and Tarski (1928).

 $^{^2}$ In the Veinott-Zhou theorem, weakly monotone means ascending in the strong set order sense.

 $^{^{3}}$ For example, see the results on constructive versions of Tarski's theorem in Cousot and Cousot (1979) and Echenique (2005). To the best of our knowledge, this paper is the first paper in the literature that provides constructive, iterative methods of finding fixed points from the Veinott-Zhou result.

⁴ In economic applications, the difference between complete lattices and σ -complete lattices can be important. One such example is a fixed-point problem in spaces of (Borel) measurable functions over a compact domain $A \subset \mathbb{R}^n$, where the space has least and greatest elements, and is endowed with a pointwise partial ordering. This space is generally only σ -complete. When this space is given an almost everywhere pointwise partial ordering, its equivalence classes become a complete lattice (e.g., see Van Zandt (2010), Lemma 5).

tinuous correspondence $F : A \rightrightarrows A$ that transforms a complete lattice (resp., a σ -complete lattice) starting from any given initial point $a^0 \in A$. More precisely, given any $a^0 \in A$, we construct fixed points \underline{a}^* and \overline{a}^* such that sufficiently remote elements a^k of any sequence of iterations $(a^k)_{k=0}^{\infty}$ (i.e., such that $a^{k+1} \in F(a^k)$ for all k) are contained between \underline{a}^* and \overline{a}^* . The fixed points \underline{a}^* and \overline{a}^* are sharp or tight, i.e., $\underline{b} \leq \underline{a}^*$ and $\overline{a}^* \leq \overline{b}$ if fixed points \underline{b} and \overline{b} are such that remote iterations of F starting at a^0 are located between \underline{b} and \overline{b} .⁵

The fixed points \underline{a}^* and \overline{a}^* are given by formulas. This allows for performing various exercises, such as comparative statics, which would be difficult if we just showed the existence of the sharp fixed-point bounds. However, the formulas involve double limits. So, their usefulness for the purpose of computation is somewhat limited. In response to this problem we provide a condition (see Remark 4), satisfied in a number of examples that we studied, under which it is enough to compute a finite number of iterations to approximate \underline{a}^* and \overline{a}^* .

One might argue that extensions of fixed-point theorems from functions to correspondences have worked pretty well in a variety of settings. In particular, in our setting one may consider iterations $a^{k+1} = \inf F(a^k)$ starting from $\inf A$ to obtain the lower fixed-point bound, and iterations $a^{k+1} = \sup F(a^k)$ starting from $\sup A$ to obtain the upper fixed-point bound. Exploring such sequences is in fact the main idea behind the Veinott-Zhou extension of Tarski's theorem. We show, however, that this idea does *not* deliver the desired extension. Firstly, these extremal selections⁶ are typically not order continuous and hence do not satisfy the assumptions of Olszewski (2021b). And secondly, even in cases when

⁵ It is important to note that little is known in the existing literature even about the existence of fixed points for monotone upper order hemicontinuous correspondences that transform σ complete lattices. Our arguments here verify the existence, as well as provide iterative formulas for tight fixed-point bounds for any initial $a^0 \in A$.

⁶ A selection of a correspondence $F : A \rightrightarrows B$ is any function $f : A \rightarrow B$ such that $f(a) \in F(a)$ for any $a \in A$.

the extremal selections are order continuous, the above procedure would deliver fixed-point bounds, but they would not necessarily be *tight*.

This paper is related to an important and large literature in economics that applies the Tarski-Kantorovich fixed-point theorem when studying the existence of equilibria.⁷ The Tarski-Kantorovich theorem says for an order continuous transformation of a countably chain complete partially ordered set (CCPO) with least (resp., greatest) elements, the supremum (resp., infimum) of iterations from the least (resp., greatest) element of the CCPO will converge in order to the least (resp., greatest) fixed point.⁸ One way of understanding the results in Olszewski (2021b) is that he shows that for order continuous functions that transform a complete lattice, there exists a generalization of the Tarski-Kantorovich theorem, where from any element of the function's domain, elements of the fixed-point set form tight bounds on sufficiently remote iterations. This paper extends this idea to the case of correspondences. In particular, we show the Tarski-Kantorovich theorem holds in both: the setting of the original Veinott-Zhou extension of Tarski's theorem for complete lattices, where the correspondence is additionally assumed to be upper order hemicontinuous, as well as in the setting of strongly monotone upper order hemicontinuous correspondences in σ -complete lattices, where the correspondence values are additionally required to have a least and a greatest element.

To obtain our results, we introduce a new notion of order continuity for monotone correspondences (i.e., "upper order hemicontinuity"). In Section 5 of this pa-

⁷ Some examples of work in economics applying the Tarski-Kantorovich theorem include papers on supermodular games (e.g., Van Zandt (2010), Kunimoto and Yamashita (2020), Balbus et al. (2022)), rationalizability in games (e.g. Ok (2004)), models of production chains (Kikuchi et al., 2018), dynamic programming with unbounded returns (e.g., Kamihigashi (2014), Becker and Rincón-Zapatero (2021) among many others), the existence of recursive equilibrium in dynamic stochastic growth models (e.g., Coleman (1991), Mirman et al. (2008), Datta et al. (2018)), computing Bewley models in macroeconomics (e.g., Li and Stachurski (2014), Açıkgöz (2018)).

⁸ For example, see Jachymski et al. (2000), Theorem 1, and Dugundji and Granas (1982), p.15 for a discussion of the Tarski-Kantorovich theorem. See also Balbus et al. (2015), Theorem 1.

per, we show the result for the upper order hemicontinuous correspondences can be extended to correspondences with discontinuities but its proof is more involved, requires transfinite constructions, and the extension is perhaps of less interest for economists (particularly in applications). The fact that such transfinite constructions are required without order continuity is not surprising given the literature on constructive characterizations of Tarski's theorem where transfinite arguments appear indispensable (e.g., Cousot and Cousot (1979) or Echenique (2005) among others).

We believe our extensions in this paper are important because most iterations that we consider in economics (and perhaps in other areas of research) use correspondences. For example, players happen to have multiple best responses in games, including those of strategic complementarities, and consumers or producers happen to have multiple optimal bundles. Multiplicity appears when payoffs are not strictly quasi-concave with respect to players' own actions, and consumers' or producers' choices. For example, a small reduction in an oligopolist's price may lower its current profits, but a larger reduction, which lowers the current profits by more, may make other firms exit or deter subsequent entry; or in a contest with multiple prizes whose values are convex, the increase in the expected value of prize induced by a small increase in effort may not be worth the cost of this additional effort, but a larger effort may result in a sufficient increase in prize to compensate for the effort cost.

To illustrate the usefulness of tight fixed-point bounds for increasing correspondences we present an application of our results to a class of social learning models on networks (studied recently by Cerreia-Vioglio et al. (2023)). Our tools allow for extending their analysis in the cases in which social learning may not result in converging to a consensus. In particular, our bounds enable us to estimate numerically the limit amount of disagreement. Despite the fact that our bounds use double limits, the iterative numerical procedure that we analyze in our example gave the bounds (up to 5 decimal places) in fewer than 25 iterations.

2 Preliminaries

We start with introducing some basic definitions. A partially ordered set (or *poset*) is set A equipped with a partial order \geq . For $a', a \in A$, we say a' is *strictly higher* than a, and write a' > a, whenever $a' \geq a$ and $a' \neq a$. A poset (A, \geq) is a *lattice* if for any $a, a' \in A$ there exist the least upper bound of $\{a, a'\}$ (denoted by $a \lor a'$ or $\sup\{a, a'\}$) and the greatest lower bound of $\{a, a'\}$ (denoted by $a \land a'$ or $\inf\{a, a'\}$). A lattice A is *complete* if there also exist $\bigvee B := \sup B \in A$ and $\bigwedge B := \inf B \in A$ for all $B \subseteq A$. A lattice A is σ -complete, whenever for any countable $B \subseteq A, \bigvee B$ and $\bigwedge B$ exist in A. A subset $B \subseteq A$ is a *sublattice* of A if $a \lor a'$ and $a \land a'$, as defined in (A, \geq) , belong to B for all $a, a' \in B$. A sublattice B of a lattice A is a *subcomplete sublattice* if for any $C \subseteq B$ the supremum $\bigvee C$ and the infimum $\bigwedge C$, as defined in (A, \geq) , belong to B.

We can compare subsets of A using set relations compatible with (A, \geq) . Let 2^A denote the set of all subsets of A. If (A, \geq) is a poset, and $B, B' \in 2^A \setminus \{\emptyset\}$, we write $B' \geq^S B$ if for all $b' \in B'$, $b \in B$, $b' \geq b$. If (A, \geq) is a lattice, B and B' are two nonempty subsets of A, we say B' is (Veinott)-strong set order higher than B, denoted by $B' \geq^{SSO} B$, whenever for any $b' \in B'$ and $b \in B, b' \wedge b \in B$ and $b' \vee b \in B'$.

Let $F : A \Longrightarrow B$ be a nonempty-valued correspondence, where (A, \ge) and (B, \ge) are posets. We say F is strongly monotone (increasing) whenever a' > a implies that $F(a') \ge^{S} F(a)$. Now, let (B, \ge) be a lattice. We say F is weakly monotone (increasing) whenever a' > a implies that $F(a') \ge^{SSO} F(a)$.

A sequence $(a^k)_{k=0}^{\infty}$ of elements of A is *increasing* if $a^{k+1} \ge a^k$ for all k. It

is strictly increasing if $a^{k+1} > a^k$ for all k. Decreasing and strictly decreasing sequences can be defined in the obvious specular manner. A monotone sequence then is either increasing or decreasing. We say that an increasing (resp., decreasing) sequence $(a^k)_{k=0}^{\infty}$ converges to $a \in A$ whenever $\bigvee_{k\geq 0} a^k = a$ (resp., $\bigwedge_{k\geq 0} a^k = a$). That is, when a is the supremum (resp., infimum) of the increasing (resp., decreasing) sequence.

Definition 1. We say that a correspondence F is upper order hemicontinuous⁹ whenever it satisfies the following condition: if a monotone sequence $(a^k)_{k=0}^{\infty}$ converges to a, then any monotone sequence $(b^k)_{k=0}^{\infty}$ such that $b^k \in F(a^k)$ for all kconverges to some $b \in F(a)$.

Clearly, when F is weakly or strongly monotone, then for any monotone sequence $(a^k)_{k=0}^{\infty}$ there exists a monotone sequence $(b^k)_{k=0}^{\infty}$ such that $b^k \in F(a^k)$ for all k, hence our definition is not vacuous.

Finally, a function $f : A \to B$ is order-preserving (or increasing) on A if $a \leq a'$ implies $f(a) \leq f(a')$ for any a, a' in A. A function f is upward order continuous (resp., downward order continuous) if for any increasing (resp., decreasing) convergent sequence $(a^k)_{k=0}^{\infty}$ with $a^k \in A$, we have:

$$f\left(\bigvee_{k\in\mathbb{N}}a^k\right) = \bigvee_{k\in\mathbb{N}}f(a^k) \quad \left(\text{resp. } f\left(\bigwedge_{k\in\mathbb{N}}a^k\right) = \bigwedge_{k\in\mathbb{N}}f(a^k)\right).$$

A function f is order continuous if it is both upward and downward order continuous. Notice, if f is upward (resp., downward) order continuous, it is order preserving or increasing.¹⁰

⁹ Our definitions coincide with standard definitions of upper hemicontinuity, when A is endowed with the order topology.

¹⁰ If a function is upward (resp., downward) order continuous, it is also sup (resp., inf) preserving (e.g., Dugundji and Granas (1982), p. 15).

3 Iterations on monotone upper order hemicontinuous correspondences

In this section, we will state and prove our results under the following two alternative sets of assumptions:

Assumption 1. A is a complete lattice. $F : A \rightrightarrows A$ is weakly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, F(a) is a subcomplete sublattice of A.

Assumption 2. A is a σ -complete lattice. $F : A \rightrightarrows A$ is strongly monotone and upper order hemicontinuous. Moreover, for any $a \in A$, the supremum and the infimum of F(a) belong to F(a).

Three comments are in order. First, in games of strategic complements (GSCs), the best-reply correspondences are weakly monotone. If additionally, the complementarities between own player actions and other players actions are strictly single crossing, the best-reply correspondences are strongly monotone. Second, upper order hemicontinuity turns out to be a natural condition that is easy to check in many economic applications. For example, in games where payoff functions are jointly order continuous in action profiles, the resulting best-reply correspondences are upper order hemicontinuous as a consequence of well-known maximum theorems (e.g., Berge's theorem). It is worth noting that upper *order* hemicontinuity is typically weaker than requiring upper hemicointuity with respect to "more standard" topology, as the former is required to hold for monotone sequences only. Third, there are settings of interest in which Assumption 2 naturally holds. In particular, GSCs where the best response correspondence maps σ -complete lattices. Examples of such games include interim formulation of Bayesian supermodular games with strict complementarities between actions or Schmeidler's formulation of large stochastic supermodular game (see e.g. Van Zandt (2010), Balbus et al. (2014) or Balbus et al. (2019)). In all these examples players are required to use measurable strategies (as functions of types or states) and hence the actions spaces are only σ -complete (when endowed with pointwise partial orders).

We will first state our main results on the existence of fixed points.¹¹ We will next introduce an iterative process that allows for attaining these fixed points by starting from any point of the lattice. We will then state and prove our main result in a somewhat stronger form that refers to this iterative process.

Proposition 1. Suppose that Assumption 1 or Assumption 2 is satisfied. For any given a point $a^0 \in A$, there exists a fixed point \underline{a}^* such that $\liminf_k a^k \geq \underline{a}^*$ for any sequence $(a^k)_{k=0}^{\infty}$, where $a^{k+1} \in F(a^k)$ for $k \geq 1$, and there exists a fixed point \overline{a}^* such that $\limsup_k a^k \leq \overline{a}^*$ for any sequence $(a^k)_{k=0}^{\infty}$, where $a^{k+1} \in F(a^k)$ for $k \geq 1$. In addition, if \underline{b} is a fixed point of F which has the above property of \underline{a}^* , then $\underline{a}^* \geq \underline{b}$, and if \overline{b} is a fixed point of F which has the above property of \overline{a}^* , then $\overline{a}^* \leq \overline{b}$.

Proposition 1 under Assumption 1 follows from the generalization of Taski's fixed-point theorem to correspondences. Indeed, one can define \underline{a}^* as the greatest fixed-point \underline{b} with the property that $\liminf_k a^k \geq \underline{b}$ for all iterative sequences $(a^k)_{k=0}^{\infty}$. However, it is not obvious that such a fixed point exists under Assumption 2. In addition, even under Assumption 1 Proposition 1 delivers only the existence of fixed-point bounds. So, we will proceed in a different way, and construct such a fixed point \underline{a}^* .¹²

The construction of a \underline{a}^* and \overline{a}^* will take a number of steps and lemmas. We

¹¹ It bears mentioning that we are not aware of any results in the literature on the existence of fixed pointS for increasing correspondences that transform σ -complete lattices. In particular, we are not aware of any results that extend any version of the Veinott-Zhou theorem (e.g., Veinott (1992) and Zhou (1994b)) to domains that are only σ -complete.

¹²A similar comment concerns \overline{a}^* .

illustrate this construction, step by step, by an example in Appendix B.

Define functions $\underline{F}: A \to A$ and $\overline{F}: A \to A$ as follows

$$\underline{F}(a) := \bigwedge F(a) \text{ and } \overline{F}(a) := \bigvee F(a).$$

Under Assumption 1, as well as Assumption 2, \overline{F} and \underline{F} are both well-defined selections of F. All lemmas below hold true under Assumption 1 as well as under Assumption 2. We will therefore not explicitly make these assumptions in the statements of the lemmas. The proofs of the lemmas are relegated to Appendix A.

Lemma 1. \overline{F} (resp., \underline{F}) is downward order continuous (resp., upward order continuous) on A.

Now, let $\underline{a}^1 = \underline{F}(a^0)$ and $\overline{a}^1 = \overline{F}(a^0)$ be the infimum and the supremum of $F(a^0)$; by induction, for k = 1, 2, ... let \underline{a}^{k+1} and \overline{a}^{k+1} be the infimum of $F(\underline{a}^k)$ and supremum of $F(\overline{a}^k)$, i.e.

$$\underline{a}^{k+1} = \underline{F}(\underline{a}^k) \quad \text{and} \quad \overline{a}^{k+1} = \overline{F}(\overline{a}^k).$$

It will be convenient to define \underline{a}^0 and \overline{a}^0 as a^0 . Let $\underline{a}^{\omega} = \liminf_k \underline{a}^k$ and $\overline{a}^{\omega} = \limsup_k \overline{a}^k$. That is,

$$\underline{a}^{\omega} = \bigvee_{k} \bigwedge_{l \ge k} \underline{a}^{l} \text{ and } \overline{a}^{\omega} = \bigwedge_{k} \bigvee_{l \ge k} \overline{a}^{l}.$$

Lemma 2. There exists $a \in F(\underline{a}^{\omega})$ such that $a \leq \underline{a}^{\omega}$ Similarly, there exists $a \in F(\overline{a}^{\omega})$ such that $a \geq \overline{a}^{\omega}$.

Under Assumption 1, let $\underline{a}^{\omega+1}$ be the supremum of the elements of $F(\underline{a}^{\omega})$ that are no grater than \underline{a}^{ω} , and let $\overline{a}^{\omega+1}$ be the infimum of the elements of $F(\overline{a}^{\omega})$ that are no smaller than \overline{a}^{ω} . That is:

$$\underline{a}^{\omega+1} = \bigvee F(\underline{a}^{\omega}) \cap I(\underline{a}^{\omega}) \quad \text{and} \quad \overline{a}^{\omega+1} = \bigwedge F(\overline{a}^{\omega}) \cap J(\overline{a}^{\omega})$$

with $I(a) := \{a' \in A : a' \leq a\}$ and $J(a) := \{a' \in A : a' \geq a\}$.

Lemma 2 guarantees that under Assumption 1, $\underline{a}^{\omega+1}$ and $\overline{a}^{\omega+1}$ are well defined because F(a), for all a, is a subcomplete sublattice. Under Assumption 2, F(a)is not necessarily a subcomplete sublattice. In this case, let $\underline{a}^{\omega+1} = \underline{a}^{\omega}$ if \underline{a}^{ω} is a fixed point, and let $\underline{a}^{\omega+1} = \inf F(\underline{a}^{\omega})$ otherwise.

We can now continue our iterations starting from \underline{a}^{ω} and \overline{a}^{ω} . We define the following sequences $(\underline{a}^{\omega+k})_{k=1}^{\infty}$ and $(\overline{a}^{\omega+k})_{k=1}^{\infty}$ recursively as follows:

$$\underline{a}^{\omega+k+1} = \bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \quad \text{and} \quad \overline{a}^{\omega+k+1} = \bigwedge F(\overline{a}^{\omega+k}) \cap J(\overline{a}^{\omega+k}).$$
(1)

under Assumption 1, and $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ if $\underline{a}^{\omega+k}$ is a fixed point, and $\underline{a}^{\omega+k+1} = \inf F(\underline{a}^{\omega+k})$ otherwise under Assumption 2.

This yields the following results:

Lemma 3. The sequences $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ and $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ are both well-defined. Moreover, if any $\underline{a}^{\omega+k_0}$ (resp., $\overline{a}^{\omega+k_0}$) is a fixed point of F, then the sequence $(\underline{a}^{\omega+k})_{k=k_0}^{\infty}$ (resp., $(\overline{a}^{\omega+k})_{k=k_0}^{\infty}$) is constant.

Lemma 4. (i) The sequence $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ is decreasing, and its limit \underline{a}^* is a fixed point of F; (ii) the sequence $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ is increasing and its limit \overline{a}^* is a fixed point of F.

This completes our construction of fixed points \underline{a}^* and \overline{a}^* . It possibly appears as a puzzling feature of the construction that \underline{a}^{k+1} is defined as the infimum of $F(\underline{a}^k)$, while $\underline{a}^{\omega+k+1}$ is defined as the supremum of $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. (A similar question concerns \overline{a}^{k+1} and $\overline{a}^{\omega+k+1}$.) For the definition of \underline{a}^{k+1} we had no choice. It had to be the infimum of $F(\underline{a}^k)$ to guarantee \underline{a}^* is indeed a lower bound for the large iterations of F. In turn, if we defined $\underline{a}^{\omega+k+1}$ as the infimum of $F(\underline{a}^{\omega+k})$, and F is only weakly increasing, then \underline{a}^* would still be a fixed-point lower bound for the iterations of F, but it could not be the sharp one. This is illustrated by the following example. (See also the example from Appendix B). **Example 1.** Consider the complete lattice

$$A = \{(0,0), (1,1), (2,2), (2,3), (3,2), (3,3)\},\$$

with $F(0,0) = F(1,1) = F(2,2) = \{(0,0), (1,1)\}, F(2,3) = \{(3,2)\}, F(3,2) = \{(2,3)\}$ and $F(3,3) = \{(3,3)\}$. Taking $a^0 = (2,3)$ we get $\underline{a}^{\omega} = (2,2)$ which is not a fixed point of F. But applying our second round of iterations we obtain a sharp fixed-point bound $\underline{a}^* = (1,1)$. If we defined instead $\underline{a}^{\omega+1} = \inf F(\underline{a}^{\omega})$, we would obtain a fixed-point bound a = (0,0) but it would not be the sharp one, however.

Note finally that under Assumption 2, $\underline{a}^{\omega+k+1}$ and $\overline{a}^{\omega+k+1}$ can be defined as any element of $F(\underline{a}^{\omega+k})$ such that $\underline{a}^{\omega+k+1} < \underline{a}^{\omega+k}$, and $\overline{a}^{\omega+k+1}$ can be defined as any element of $F(\overline{a}^{\omega+k})$ such that $\overline{a}^{\omega+k+1} > \overline{a}^{\omega+k}$, unless $\underline{a}^{\omega+k}$ (resp., $\overline{a}^{\omega+k}$) is a fixed point, in which case $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ (resp., $\overline{a}^{\omega+k+1} = \overline{a}^{\omega+k}$). This is so because for strongly monotone F, if $\underline{a}^{\omega+k}$ is not a fixed point of F, then $\sup F(a) \leq \inf F(\underline{a}^{\omega+k})$ for all $a \in F(\underline{a}^{\omega+k})$ such that $a < \underline{a}^{\omega+k}$. So, no element of $F(\underline{a}^{\omega+k})$ can be a fixed point possibly except $\inf F(\underline{a}^{\omega+k})$. Thus, by defining $\underline{a}^{\omega+k+1}$ in the alternative way, we can be sure that we will not "jump down" over any fixed point.

We can now state and prove the following key result.

Proposition 2. Both under Assumption 1 and under Assumption 2, the following statements hold true: (i) for any given $a^0 \in A$, \underline{a}^* has the property that $\liminf_k a^k \geq \underline{a}^*$ for all iterative sequences $(a^k)_{k=0}^{\infty}$, and \overline{a}^* has the property that $\limsup_k a^k \leq \overline{a}^*$ for all iterative sequences $(a^k)_{k=0}^{\infty}$.

(ii) Suppose that \underline{b} is a fixed point of F such that for any sequence $(a^k)_{k=0}^{\infty}$, where $a^{k+1} \in F(a^k)$ for $k \ge 1$, we have $\liminf_k a^k \ge \underline{b}$. Then $\underline{a}^* \ge \underline{b}$. Suppose that \overline{b} is a fixed point of F such that for any sequence $(a^k)_{k=0}^{\infty}$, where $a^{k+1} \in F(a^k)$ for $k \ge 1$, we have $\limsup_k a^k \le \underline{b}$. Then $\overline{a}^* \le \overline{b}$.

Proof. We will prove the theorem for \underline{a}^* ; the proof for \overline{a}^* is analogous. Part (i) follows directly from the definitions and our construction. We will prove part (ii).

Since $\liminf_k a^k \geq \underline{b}$ thus, $\underline{b} \leq \underline{a}^{\omega}$. This completes the proof if $\underline{a}^* = \underline{a}^{\omega}$. If not, then \underline{a}^{ω} is not a fixed point and $\underline{b} < \underline{a}^{\omega}$. Recall $\underline{b} \in F(\underline{b})$, and $\underline{a}^{\omega+1} \in F(\underline{a}^{\omega})$. Under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^{\omega})$ because $F(\underline{b}) \leq^{SSO} F(\underline{a}^{\omega})$. Since $\underline{b} < \underline{a}^{\omega}$, and by the definition of $\underline{a}^{\omega+1}$ and because \underline{a}^{ω} is not a fixed point, $\underline{a}^{\omega+1} < \underline{a}^{\omega}$, we have that $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega}$. This implies that $\underline{b} \vee \underline{a}^{\omega+1} \in F(\underline{a}^{\omega}) \cap I(\underline{a}^{\omega})$. Since $\underline{a}^{\omega+1}$ is the greatest element of this set, hence $\underline{b} \vee \underline{a}^{\omega+1} \leq \underline{a}^{\omega+1}$. So $\underline{b} \leq \underline{a}^{\omega+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+1}$ because $\underline{b} < \underline{a}^{\omega}$, so the strong monotonicity of F implies $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^{\omega}) = \underline{a}^{\omega+1}$. We will show $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and consequently $\underline{b} \leq \underline{a}^*$. We have proven this for k = 1; suppose it is the case for some k. The proof is complete if $\underline{a}^{\omega+k}$ is a fixed point, because $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$. If $\underline{a}^{\omega+k}$ is not a fixed point, $\underline{b} < \underline{a}^{\omega+k}$, and $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$. Moreover, $\underline{b} \vee \underline{a}^{\omega+k+1} \in I(\underline{a}^{\omega+k})$, hence $\underline{b} \vee \underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Since $\underline{a}^{\omega+k+1}$ was defined as the greatest element of this set under Assumption 1, $\underline{b} \vee \underline{a}^{\omega+k+1} \leq \underline{a}^{\omega+k+1}$, consequently $\underline{b} \leq \underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{b} \leq \underline{a}^{\omega+k+1}$ because $\underline{b} < \underline{a}^{\omega+k}$, so strong monotonicity of F implies that $\underline{b} \leq \sup F(\underline{b}) \leq \inf F(\underline{a}^{\omega+k}) = \underline{a}^{\omega+k+1}$. Thus, $\underline{b} \leq \underline{a}^{\omega+k}$ for any k, and also $\underline{b} \leq \underline{a}^*$.

Proposition 2 captures formally the intuition that \underline{a}^* and \overline{a}^* are tight fixedpoint bounds between which sufficiently large iterations of F are located.

Remark 1. Our Propositions 1 and 2 recover the central results in Olszewski (2021b) and improve upon them from the case of strongly monotone upper order hemicontinuous correspondences. Recall that he studied the fixed-point bounds generated by remote iterations of order continuous (hence monotone) function $f: A \to A$ from any initial element $a^0 \in A$, where A is a complete lattice.

Thus, if in a GSC where best replies are correspondences, the results of Olszewski (2021b) are not applicable, unless, e.g., the least and greatest best replies are continuous function.¹³ This is a key limitation especially in economic appli-

 $^{^{13}}$ For a weakly monotone upper order hemicontinuous correspondence, in general, order con-

cations.¹⁴ However, one way of interpreting our Lemma 1 is that the existence of fixed-point bounds can be generalized to correspondences if one observes that the greatest (resp., the least) selections is downward order continuous (resp., upward order continuous), because downward (resp., upward) order continuity is sufficient for the existence of upper (resp., lower) bound.

Finally, for strongly monotone correspondences, we allow the domain A to be only σ -complete lattice, while Olszewski (2021b) requires A to be a complete lattice. This distinct can be important in applications as the Remarks 2 and 3 below note.

Remark 2. Lemma 1 is of independent interest. Lemma 1 remains true under weaker conditions than those in Assumption 1. In particular, if A is σ -complete lattice and F is weakly monotone, upper order hemicontinuous with $\bigvee F(a) \in A$ and $\bigwedge F(a) \in A$, the statement of Lemma 1 remains valid. This then immediately implies F has a least and greatest fixed point, if A has the greatest and least elements, by an application of the following version of the standard Tarski-Kantorovich theorem: if A is a countably chain complete poset (in our case here, A is σ -complete) with a least element <u>a</u> and a greatest element <u>a</u> and $f : A \to A$ is upward order continuous function, then f has a least fixed point: $\underline{a}^* = \sup_k \{f^k(\underline{a})\}$ and if $f: A \to A$ is downward order continuous, then f has a greatest fixed point: $\overline{a}^* = \inf_k \{f^k(\overline{a})\}$. A natural way to generalize this theorem to a correspondence $F: A \rightrightarrows A$ is to consider iterations of its least selection and its greatest selection from the least and greatest elements of A respectively. If the least selection exists and is upward order continuous, then F has a least fixed point $\underline{a}^* = \sup_k \{\underline{F}^k(\underline{a})\}$. Similarly, if the greatest selection exists and is downward order continuous, then F has a greatest fixed point $\overline{a}^* = \inf_k \{\overline{F}^k(\overline{a})\}$. This implies, for example, that Lemma 1 can be applied to obtain the main results in Van Zandt (2010) on the

tinuous selectors need not exist.

¹⁴ See examples of GSC, where best replies are not single-valued in Topkis (1979), Vives (1990) or Milgrom and Shannon (1994).

existence of least and greatest interim Bayesian Nash Equilibrium (BNE) for the class of supermodular games of incomplete information he studies (e.g., prove his existence result in Theorem 10).¹⁵

Remark 3. Proposition 1 can be used to construct an iterative characterization of the Bayesian Nash Equilibrium (BNE) in the class of interim Bayesian supermodular games studied in Van Zandt (2010) if in addition to all the basic assumptions in Van Zandt (2010), we assume each player has strict increasing differences between their own actions and other players actions. Then the resulting joint best reply correspondence will be strongly monotone and satisfy Assumption 2. If, as in Van Zandt (2010), for the case of interim BNE, the domain for the joint best reply correspondence is taken to be the (σ -complete) set of measurable functions mapping types to actions endowed with the standard pointwise partial ordering, Proposition 1 (and also Proposition 3 from Section 5) then imply the existence of not only least and greatest BNE, but also tight "local" least and greatest BNE bounds for iterations starting from any initial profile of (measurable) strategies.

Remark 4. Our paper provides a constructive way of deriving fixed points for the correspondence F. That is, for each initial a^0 we propose a sequence $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ that converges to the fixed-point bound \underline{a}^* (and similarly a sequence $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ that converges to the fixed-point bound \overline{a}^*). The algorithmic procedures to compute these fixed-point bounds are limited, however, by the fact that to compute \underline{a}^{ω} (and \overline{a}^{ω}) one needs to compute limits of sequences $(\underline{a}^k)_{k=0}^{\infty}$ (and similarly $(\overline{a}^k)_{k=0}^{\infty}$), which, in general, requires to perform infinitely many operations already. So, we typically must stop at some k, and use \underline{a}^k or \overline{a}^k as an approximation of \underline{a}^{ω} or \overline{a}^{ω} . In practice, this comes at the risk of stopping the iterations of \underline{F} too early and starting the second iterations too far from \underline{a}^{ω} (the same is true for the upper iterations).

¹⁵ We can also obtain his existence for greatest ex-ante BNE (e.g., his corollary 8) but for the case that a space of measurable functions is ordered pointwise (and not a.e. pointwise ordered as in his construction and in Vives (1990) (lemma 6.1 and Theorem 6.1)).

While, this can be the case in general, here we provide a condition under which such too early stopping of the first iterations is avoided. We provide it for the lower iterations, but analogous condition and reasoning can be provided for upper iterations. Suppose, there exists k and n such that $\underline{a}^{n+k} = \underline{F}(\underline{a}^n)$. Then \underline{a}^n is a fixed point of the k-th iteration of the function \underline{F} . Then, we obtain:

$$\underline{a}^{\omega} = \inf\{\underline{a}^n, \dots, \underline{a}^{n+k-1}\},\$$

and it is sufficient to compute a finite number of elements to compute \underline{a}^{ω} . Our condition is satisfied, for instance, in Example 1, where $a^0 = (2,3)$. But see also Appendix B for an example, where such an early stopping is a problem for computing the tight fixed-point lower bound.

4 Application: Social learning on networks

DeGroot's model, in which agents take weighted averages of the opinions they observe, is a commonly applied approach to studying social learning on networks. Obviously, this very specific type of learning cannot well describe all real-life situations of interest. Cerreia-Vioglio et al. (2023) recently suggested a more general model, in which an opinion aggregator is a function that satisfies certain axioms. In their model of an economy of n agents, an opinion profile is represented by a vector $a \in [0, 1]^n$, and learning is represented by an opinion aggregator $T : [0, 1]^n \to [0, 1]^n$ that is monotone¹⁶ with respect to coordinate-by-coordinate ordering on $[0, 1]^n$.

We now illustrate the usefulness of our results by applying them to the setting studied by Cerreia-Vioglio et al. (2023). A group of agents $N = \{1, 2, 3\}$ share their opinions $a^0 \in [0, 1]^3$. The weights assigned to the other agents are represented

¹⁶ In addition to monotonicity, they impose two other axioms: normalization (T(k, ..., k) = (k, ..., k) for all $k \in [0, 1]$) and translation invariance $(T(x_1 + k, ..., x_n + k) = T(x_1, ..., x_n) + (k, ..., k)$ whenever it makes sense). They all are satisfied in our application.

by the matrix:

$$W = \left[\begin{array}{rrrr} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{array} \right].$$

That is, the entries in row *i* of the matrix represent the weights assigned by agent *i* to the opinions of all agents. The average aggregator $T^1(a)$ is defined, agent by agent, as the sum of opinions multiplied by their weights. For example, agent 1 assigns weight 0.4 to her own opinion and weight 0.3 to the opinion of each other agent. So, if $a^0 = (0.8, 0.6, 0.4)$, then

$$a^{1} = T^{1}(a^{0}) = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.6 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.62 \\ 0.5 \\ 0.56 \end{bmatrix}.$$

The aggregation through weighted averages satisfies the conditions required by Cerreia-Vioglio et al. (2023). It also satisfies Assumptions 1 and 2 of our paper. It always achieves limit consensus. For example, if $a^0 = (0.8, 0.6, 0.4)$, then it can be computed numerically that the long run opinion of all agents is 0.54286.

Consider now the aggregation induced by the median. This operator is defined as follows: the opinion of each agent in a vector of opinions a is assigned a probability equal to the agent's weight. This determines a probability distribution over opinions. $T^2(a)$ is defined, agent by agent, as the median of this distribution. For example, $a^1 = T^2(a^0) = (0.6, 0.4, 0.6)$ for $a^0 = (0.8, 0.6, 0.4)$. The median aggregator also satisfies the conditions required by of Cerreia-Vioglio et al. (2023). Actually, both weighted averages and medians were used as examples in their paper. Except the non-generic matrices, such that the sum of a proper subset of entries in some row is equal to 0.5, the median is always unique and the median aggregator is continuous and satisfies our Assumptions 1 and 2. For the remaining non-generic matrices, the median aggregator is upper order hemicontinuous and satisfies our Assumption $1.^{17}$

When $a^0 = (0.8, 0.6, 0.4), a^k = (0.6, 0.4, 0.6)$ for odd k and $a^k = (0.6, 0.6, 0.4)$ for even k. Thus,

$$\lim \inf_{k=\infty} a^k = (0.6, 0.4, 0.4) \text{ and } \limsup_{k=\infty} a^k = (0.6, 0.6, 0.6).$$

Despite the fact that no limit consensus is reached, by looking at lim inf and lim sup we can say that the limit disagreement is only of size 0.2, that is, it is only a half of the initial disagreement.

Note that while $\limsup_{k=\infty} a^k$ is a fixed point of the aggregator T^2 (namely \overline{a}^*) $\liminf_{k=\infty} a^k$ is not a fixed point; indeed, $T^2(0.6, 0.4, 0.4) = (0.4, 0.4, 0.4)$. Therefore applying the second rounds of iterations of T^2 to limit is necessary (not necessary for lim sup) if the objective is to find the fixed-point bound (namely \underline{a}^*). The fixed-point bound will be useful when a modeler believes that a stable outcome should be reached and allows for a broader class of adaptive dynamics.¹⁸

Up to now the aggregators considered in this application were functions. However, in some settings correspondence aggregators seem more appropriate. Let:

$$W = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.7 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}$$

and let the initial profile be $a^0 = (0.8, 0.4, 0.6)$. Suppose now that each agent ignores the extremal 20% of the opinions and so updates its opinion somewhere in the interval between 0.2 and 0.8 percentile of the opinion distribution. The operator in this example is upper order hemicontinuous, because so are the percentiles. So our Assumption 1 is satisfied.

¹⁷For all $a, T^2(a)$ is a product of intervals.

¹⁸ An aggregator that takes as input only the most recent opinions generate an extreme form of adaptive learning. If the median was applied to the minimum of opinions of each agent in two previous periods, we would obtain the convergence to the consensus (0.4, 0.4, 0.4). We postpone more detailed analysis of other forms of adaptive learning for future research. However, it seems that our fixed points are the right bounds for richer classes of adaptive learning processes.

Computing the upper iterations, i.e. iterating on the 0.8 percentile aggregator, we immediately obtain that $\overline{a}^k = (0.8, 0.6, 0.6)$ for any $k \ge 1$ and, although no consensus is reached, a profile (0.8, 0.6, 0.6) is the upper fixed-point bound of our learning process.

For the lower iterations, we obtain: $\underline{a}^k = (0.6, 0.6, 0.4)$ for odd k and $\underline{a}^k = (0.6, 0.4, 0.6)$ for even k. The learning cycles, but

$$\lim \inf_{k=\infty} \underline{a}^k = (0.6, 0.4, 0.4).$$

Now:

$$T(\liminf_{k=\infty} \underline{a}^k) = [0.4, 0.6] \times [0.4, 0.6] \times [0.4, 0.6].$$

Interestingly, both: the least element of $T(\liminf_{k=\infty} \underline{a}^k)$, namely (0.4, 0.4, 0.4), as well as its greatest element, namely (0.6, 0.6, 0.6), are fixed points of T. But the tight lower fixed-point bound is (0.6, 0.4, 0.4). That is in line with our construction on pages 10-11. The disagreement between both sharp fixed-point bounds is hence 0.2.

5 Iterations on discontinuous correspondences

One may wonder whether our results apply to discontinuous weakly monotone correspondences. The answer to these questions is negative, even for functions, as the following example shows.

Example 2. Let $A = [0,1) \cup \{2 - 1/k : k = 1,2,...\} \cup \{2,3\}$ with the lattice structure inherited from the reals. Let $f : A \to A$ be given by f(a) = a for a from [0,1), f(a) = 2 - 1/(k+1) for a = 2 - 1/k, and f(a) = 3 for a = 2,3. Points a < 1 and a = 3 are the fixed points of function f. For $a^0 = 1$, the sequence of finite iterations $a^k = f^k(a^0) = 2 - 1/(k+1)$ is increasing and converges to a = 2, but a = 2 is not the fixed point of f. So, $\overline{a}^{\omega+1} = 3$; more generally, $\overline{a}^{\omega+k} = 3$

for all $k \ge 2$. The tight fixed-point upper bound for this sequence of iterations is a = 3. Interestingly, [0, 1) is a set of all fixed-points of f that are lower than a^0 and all its iterations, but the supremum of this set is not a fixed point of f. So, none of the points from [0, 1) is a tight fixed-point lower bound.

In fact, Proposition 2 can be extended to discontinuous weakly monotone correspondences, but the cost of relaxing our continuity condition is that we must introduce transfinite iterations. In addition, we must restrict attention to iterating correspondences that transform complete (not σ -complete) lattice A. More precisely, the following result can be obtained by modifying the proof from Olszewski (2021a).

Let $\alpha > |A|$, where |A| stands for the cardinality of A, be a cardinal number. For every $\underline{a}_0 = \overline{a}^0 = a^0 \in A$, and every weakly monotone correspondence $F : A \Rightarrow A$, say that $(a_\beta)_{\beta < \alpha}$ is a sequence of transfinite iterations of F if:

$$a_{\beta} \in F(a_{\beta-1})$$
 if β has a predecessor $\beta - 1$;

and

$$\bigvee_{\gamma < \beta \gamma \le \delta < \beta} a_{\delta} \le a_{\beta} \le \bigwedge_{\gamma < \beta \gamma \le \delta < \beta} a^{\delta} \text{ if } \beta \text{ is a limit ordinal.}$$

In addition, distinguish two special sequences of transfinite iterations

$$\underline{a}^{\beta} =: \begin{cases} \inf F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta-1 \\ \bigvee_{\gamma < \beta \gamma \le \delta < \beta} a^{\delta} \text{ if } \beta \text{ is a limit ordinal.} \end{cases}$$
(2)

and

$$\overline{a}^{\beta} =: \begin{cases} \sup F(a^{\beta-1}) \text{ if } \beta \text{ has a predecessor } \beta-1 \\ \bigwedge_{\gamma < \beta\gamma \le \delta < \beta} a^{\delta} \text{ if } \beta \text{ is a limit ordinal.} \end{cases}$$
(3)

Proposition 3. Suppose that (A, \leq) is a complete lattice, and $F : A \Rightarrow A$ is a weakly monotone correspondence such that F(a) has the smallest and the greatest element for all $a \in A$. Let $\alpha > |A|$ be a regular cardinal number.¹⁹ Then, for any $a_0 = a^0 \in A$, there exist $\underline{\beta}, \overline{\beta} < \alpha$ such that $\underline{a}_{\beta} = \underline{a}_{\underline{\beta}}$ for all $\underline{\beta} \leq \beta < \alpha$, and $\overline{a}^{\beta} = \overline{a}^{\overline{\beta}}$ for all $\overline{\beta} \leq \beta < \alpha$. In particular, $\underline{a}_{\underline{\beta}}$ and $\overline{a}^{\overline{\beta}}$ are fixed points of F.

Moreover, $\underline{a}_{\underline{\beta}}$ is the greatest fixed point \underline{a} of F with the property that $\underline{a} \leq a_{\beta}$ for sufficiently large $\beta < \alpha$ and for all sequences of transfinite iterations $(a_{\beta})_{\beta < \alpha}$, and $\overline{a}^{\overline{\beta}}$ and the smallest fixed point \overline{a} of F with the property that $a^{\beta} \leq \overline{a}$ for sufficiently large $\beta < \alpha$ and for all sequences of transfinite iterations $(a_{\beta})_{\beta < \alpha}$.

Example 2 (continued) So, what is the fixed-point lower bound from Proposition 3, for $a^0 = 1$ from Example 2? In this case $\underline{a}^{\omega} = 2$ and $\underline{a}^{\omega+k} = 3$ for $k \ge 1$. Thus, the lower bound is equal to 3. As claimed in Proposition 3, the sufficiently large transfinite iterations of a^0 are "no smaller" that this lower bound. In this case, these are all iterations from $(\omega + 1)$ -st one on. This is different from what we required in other sections, where a lower bound was supposed to be "no greater" than sufficiently large "finite" iterations of a^0 .

It is possible to obtain a somewhat stronger result than Proposition 3, which requires a somewhat more involved proof. However, since transfinite sequences are unlikely to be of interest for economists, we will not present and discuss this result in this paper.

6 Conclusion and discussion

This paper provides a generalization of the Tarski-Kantorovich theorem in Olszewski (2021b) for order continuous functions in complete lattices to the case of

¹⁹ A regular cardinal number α is defined by the following property: No set of cardinality α can be represented as the union of a family of subsets such that each subset from the family has a cardinality smaller than α , and the family itself is of a cardinality smaller than α .

weakly (resp., strongly) monotone upper order hemicontinuous correspondences in complete (resp., σ -complete) lattices. In particular, we provide least and greatest fixed-point bounds for remote iterations on the weakly (strongly) monotone correspondences from any initial element of the domain. Our results are applicable in many cases of monotone fixed-point problems for correspondences that arise naturally in various areas of economics and game theory including games of strategic complementarities, social learning, monotone Markov processes, dynamic equilibrium theory, among other applications.

In addition, the paper contributes to the literature on existence of fixed points for monotone correspondences. We show the existence of the least and greatest fixed-point bounds even in the case of weakly monotone upper order hemicontinuous correspondences that transform a σ -complete lattice. This is a significant contribution, because in many applications using monotone methods in economics research the domain of a correspondence is not a complete but only a σ -complete lattice (e.g., this is the case for correspondences defined in spaces of measurable functions under standard pointwise partial orders as in Bayesian supermodular games), and our results generalize theorems of Veinott (1992) and Zhou (1994b) from complete lattices to only σ -complete lattices.

Many interesting questions remain. From the perspective of existence for the case of weakly monotone upper order hemicontinuous correspondences, the critical extension to be considered is in what sense Propositions 1 and 2 are available when the space A is only σ -complete. In this case, the critical step that needs to be generalized is the definition of updated iterations for $\underline{a}^{\omega+k+1}$ and $\overline{a}^{\omega+k+1}$ in equation (1) for the σ -complete case. If doable, such an extension of our main result would then provide a significant extension of the Veinott-Zhou theorem and have many direct applications (e.g., to interim Bayesian Nash Equilibrium in Van Zandt (2010)).

Moreover, recently Sabarwal (2023a,b) has proposed new approaches to comparing nonempty subsets of a complete lattice (in particular star lattice set order), which may be used to define new notions of monotone correspondences. One might ask, if our constructions can be extended to these new notions of monotonicity.

Another important extension of the results of this paper is to explore the comparative statics of fixed-point bounds in Proposition 1 for parameterized monotone correspondences relative to ordered changes in the parameters. For example, one can use the classical Tarski-Kantovorich theorem for order continuous functions in countable complete partially ordered sets to compare least and greatest fixed points for different parameters (see Balbus et al. (2015)). One might ask if similar fixed-point comparative statics result relative to tight "local" fixed-point bounds are available for order continuous functions and monotone correspondences in complete or σ -complete lattices.

For parameterized weakly monotone correspondences in complete lattices, the comparative statics of extremal selectors is well-known (e.g., see Veinott (1992) and Topkis (1998)). The series of papers by Echenique (2002, 2003) study the comparative statics of non-extremal equilibria by developing a generalization of the correspondence principle of Samuelson (1947). Moreover, the already mentioned papers of Sabarwal (2023a,b) present some new results for fixed-point comparative statics. On top of valuable insights, there are limitations of this existing literature. First, all these papers require underlying domains of monotone correspondences to be complete lattices. Second, the work on the correspondence principle additionally does not apply to situation where there is a continuum of equilibria, equilibria are unstable, or adaptive dynamic adjustment processes are not convergent. Finally, with respect to the work of Sabarwal (2023a,b), his tight approximations of fixed points and tight fixed-point comparative statics is existential and not constructive (in the sense used in our paper).

Propositions 1 and 2 offer new possibilities for extensions of these existing works. In particular, our iterative methods offer the possibility of constructing "dynamic" or "iterative" comparative statics, in which we compare an equilibrium for an original parameter to the equilibrium bounds for iterations induced by a parameter change (in particular, for non-extremal equilibrium). This could allow one to extend the results of Echenique (2002, 2003) to settings were the identification of comparative statics of a given equilibrium is not possible because this equilibrium is either not stable or not locally unique. Similarly, with respect to Sabarwal (2023a,b), our methods could potentially *compute* tight fixed-point bounds after the parameter change, for the correspondences studied in his works. Such extensions of the results of this paper are being pursued in Balbus et al. (2024).

A Proofs

Proof of Lemma 1. Since F is weakly increasing, so are \overline{F} and \underline{F} . Indeed, if a' < a'' then $\underline{F}(a') \land \underline{F}(a'') \in F(a')$. As a result,

$$\underline{F}(a') \le \underline{F}(a') \land \underline{F}(a'').$$

Hence $\underline{F}(a') = \underline{F}(a') \wedge \underline{F}(a'') \leq \underline{F}(a'')$. If F is strictly increasing, then $\underline{F}(a') \leq \overline{F}(a') = \sup F(a') \leq \inf F(a'') = \underline{F}(a'')$. Similarly, we show that \overline{F} is increasing.

We prove the upward continuity of \underline{F} . The proof is the same under Assumption 1 and under Assumption 2. Let $(a^k)_{k=1}^{\infty}$ be an increasing sequence in A such that $a = \bigvee_{k \in \mathbb{N}} a^k$. Let $b^k := \underline{F}(a^k)$. Then $b^k \in F(a^k)$ for all $k \in \mathbb{N}$, and $(b^k)_{k=1}^{\infty}$ is increasing. Let $b := \bigvee b^k$. Since b^k belongs to $F(a^k)$ and the sequence $(b^k)_{k=1}^{\infty}$ is increasing, b belongs to F(a) by upper hemicontinuity of F. Hence, $\underline{F}(a) \leq b$. On the other hand, $\underline{F}(a) = \underline{F}(\bigvee_{k \in \mathbb{N}} a^k) \geq b^k$ for any k because \underline{F} is increasing. Hence $b \leq \underline{F}(a)$. Together with $\underline{F}(a) \leq b$, we have $b = \underline{F}(a)$, and hence the upward

continuity. We omit a similar proof that \overline{F} is downward continuous.

Proof of Lemma 2. We will prove the lemma for \underline{a}^{ω} ; the proof for \overline{a}^{ω} is analogous. The sequence $\left(\bigwedge_{l\geq k} \underline{a}^l\right)_{k=0}^{\infty}$ is increasing, and \underline{a}^{ω} is its supremum. Let $b^k = \underline{F}\left(\bigwedge_{l\geq k} \underline{a}^l\right)$. By Lemma 1, \underline{F} is an increasing, upward continuous function, hence $(b^k)_{k=1}^{\infty}$ is increasing as well. In addition,

$$a:=\bigvee_{k\in\mathbb{N}}b^k=\underline{F}(\underline{a}^\omega)\in F(\underline{a}^\omega).$$

To finish the proof, we must show that $a \leq \underline{a}^{\omega}$. Since $\bigwedge_{l \geq k} \underline{a}^{l} \leq \underline{a}^{l}$ for all $l \geq k$, we have that $b^{k} \leq \underline{a}^{l+1}$ for all $l \geq k$ by the monotonicity of \underline{F} and the definition of \underline{a}^{l+1} and b^{k} . So, $b^{k} \leq \bigwedge_{l \geq k+1} \underline{a}^{l} \leq \underline{a}^{\omega}$, which gives that $a = \lim_{k} b^{k} \leq \underline{a}^{\omega}$.

Proof of Lemma 3. We will show the hypothesis for the sequence $(\underline{a}^{\omega+k})_{k=0}^{\infty}$; the proof for the sequence $(\overline{a}^{\omega+k})_{k=0}^{\infty}$ is analogous. That is, we will show by induction that $\underline{a}^{\omega+k+1}$ is well-defined for any $k \ge 0$, and if $\underline{a}^{\omega+k}$ is a fixed point, then $\underline{a}^{\omega+k+1} = a^{\omega+k}$.

For k = 0, this holds true by Lemma 2. First, suppose that $\underline{a}^{\omega+k}$ is a fixed point of F for some k > 0. Then $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$, so $\underline{a}^{\omega+k+1}$ is well-defined by Assumption 1. In addition, $\underline{a}^{\omega+k+1}$ must be $\bigvee F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k})$. Hence $\underline{a}^{\omega+k+1} = \underline{a}^{\omega+k}$ by the definition of $\underline{a}^{\omega+k+1}$. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as $a^{\omega+k}$.

Suppose now that $\underline{a}^{\omega+k}$ is not a fixed point of F. By induction hypothesis $\underline{a}^{\omega+k-1}$ is neither a fixed point of F, because then $\underline{a}^{\omega+k} = \underline{a}^{\omega+k-1}$ would also be a fixed point. Hence $\underline{a}^{\omega+k-1} > \underline{a}^{\omega+k}$. By Assumption 1, $F(\underline{a}^{\omega+k}) \leq^{SSO} F(\underline{a}^{\omega+k-1})$. Take any $a' \in F(\underline{a}^{\omega+k})$. Such an a' exists because F is non-empty valued. Since

 $\underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k-1})$, it must be that $a' \wedge \underline{a}^{\omega+k} \in F(\underline{a}^{\omega+k})$ and obviously $a' \wedge \underline{a}^{\omega+k} \in I(\underline{a}^{\omega+k})$. As a result $F(\underline{a}^{\omega+k}) \cap I(\underline{a}^{\omega+k}) \neq \emptyset$. Thus, $\underline{a}^{\omega+k+1}$ is well-defined. Under Assumption 2, $\underline{a}^{\omega+k+1}$ is defined as inf $F(\underline{a}^{\omega+k})$. So, $\underline{a}^{\omega+k+1}$ is well-defined.

Proof of Lemma 4. We will prove this lemma for \underline{a}^* ; the proof for \overline{a}^* is analogous. By construction, $(\underline{a}^{\omega+k})_{k=0}^{\infty}$ is a decreasing sequence. Let \underline{a}^* be its limit. Since $\underline{a}^{\omega+k+1} \in F(\underline{a}^{\omega+k})$ for all k, by taking a limit as $k \to \infty$ and applying the upper hemicontinuity of F we obtain $\underline{a}^* \in F(\underline{a}^*)$.

B Additional example

In this appendix, we will illustrate the construction of points \underline{a}^* and \overline{a}^* with the following example. Let A be a lattice of elements from \mathbb{R}^2 with the coordinate-by-coordinate order. A is the union of three sets:

$$A_1 = \{(3,3)\} \cup \{(3-1/n, 2+1/n) : n = 1, 2, ...\};$$

 $A_2 = \{(3 - 1/n, 2) : n = 1, 2, ...\} \cup \{(3 - 1/n, 3/2) : n = 1, 2, ...\} \cup \{(3, 1 + 1/n) : n = 1, 2, ...\};$ and

$$A_3 = \{(3 - 1/n, 1) : n = 1, 2, \dots\} \cup \{(3, 1)\} \cup \{(3 - 1/n, 1/2) : n = 1, 2, \dots\} \cup \{(3, 1/2)\}.$$

See Figure 1. (Note that A is not a sublattice of \mathbb{R}^2 , it just inherits the order of \mathbb{R}^2). The correspondence F is defined by letting:

$$F(3-1/n, 2+1/n) = (3-1/(n+1), 2+1/(n+1))$$
 and $F(3,3) = (3,3)$

on A_1 ;

$$F(3-1/n,2) = \{(3-1/(n+1),3/2), (3-1/(n+1),1/2)\},\$$

$$F(3 - 1/n, 3/2) = \{(3 - 1/(n+1), 1), (3 - 1/(n+1), 1/2)\},\$$

$$F(3, 1 + 1/n) = \{(3, 1 + 1/(n+1)), (3, 1/2)\}$$

on A_2 ; and

$$F(3 - 1/n, 1) = \{(3 - 1/n, 1), (3 - 1/n, 1/2)\}$$
$$F(3, 1) = \{(3, 1), (3, 1/2)\},$$
$$F(3, 1/2) = (3, 1/2)$$

on A_3 . See again Figure 1. It is easy to check that A and F satisfy Assumption 1. Figure 1 also depicts \underline{F} , which assigns the first points from the definition of F (whenever multiplicity arises), and \overline{F} which assigns the second points. To illustrate our construction, let $a^0 = (2, 3)$. Then $\underline{a}^k = (3 - 1/(k+1), 2 + 1/(k+1))$ for k = 0, 1, ...,

$$\bigwedge_{l \ge k} \underline{a}^l = (3 - 1/(k+1), 2),$$

therefore

$$\underline{a}^{\omega} = \lim_{k} (3 - 1/(k+1), 2) = (3, 2).$$

Further, $\bigvee I(\underline{a}^{\omega}) \cap F(\underline{a}^{\omega}) = (3, 3/2)$ and $\underline{a}^{\omega+1} = (3, 3/2)$; more generally, $\bigvee I(\underline{a}^{\omega+k}) \cap F(\underline{a}^{\omega+k}) = (3, 1+1/(k+1))$ and $\underline{a}^{\omega+k+1} = (3, 1+1/(k+1))$. Thus,

$$\underline{a}^* = \lim_k \underline{a}^{\omega+k} = (3,1)$$

Note that this example illustrates a similar point to that from Example 1. If we defined $\underline{a}^{\omega+k+1}$ as the infimum of $F(\underline{a}^{\omega+k})$, then $\underline{a}^* = (3, 1/2)$ would still be a fixed-point lower bound for the iterations of F, but it could not be the sharp one.

Moreover, as argued in Remark 4, in this example one cannot stop iterating sequence $(\underline{a}^k)_{k=0}^{\infty}$ at some finite point, say n + k, to compute the tight fixed-point bound (i.e. (3, 1)) using our method. Indeed, suppose one stops iterations at \underline{a}^{n+k} and uses: $\inf{\{\underline{a}^n, \ldots, \underline{a}^{n+k-1}\}} = (3 - \frac{1}{n+1}, 2 + \frac{1}{n+1})$ as an approximation of \underline{a}^{ω} . Then one obtains $\underline{a}^* = (3, 2)$, which is even not a fixed point of F.



Figure 1: Lattice A and correspondence F from example in Appendix B. Blue arrows denote \overline{F} while red arrows denote \underline{F} (whenever relevant to distinguish).

References

- AÇIKGÖZ, O. T. (2018): "On the existence and uniqueness of stationary equilibrium in Bewley economies with production," *Journal of Economic Theory*, 173, 18–55.
- BALBUS, L., P. DZIEWULSKI, K. REFFET, AND L. WOŹNY (2019): "A qualitative theory of large games with strategic complementarities," *Economic Theory*, 67, 497– 523.
- (2022): "Markov distributional equilibrium dynamics in games with complementarities and no aggregate risk," *Theoretical Economics*, 17, 725–762.
- BALBUS, Ł., W. OLSZEWSKI, K. REFFETT, AND Ł. WOŹNY (2024): "Iterative monotone comparative statics," MS.
- BALBUS, Ł., K. REFFETT, AND Ł. WOŹNY (2014): "A constructive study of Markov equilibria in stochastic games with strategic complementarities," *Journal of Economic Theory*, 150, 815–840.
- (2015): "Time consistent Markov policies in dynamic economies with quasihyperbolic consumers," *International Journal of Game Theory*, 44, 83–112.
- BECKER, R. A. AND J. P. RINCÓN-ZAPATERO (2021): "Thompson aggregators, Scott continuous Koopmans operators, and least fixed point theory," *Mathematical Social Sciences*, 112, 84–97.
- CERREIA-VIOGLIO, S., R. CORRAO, AND G. LANZANI (2023): "Dynamic opinion aggregation: long-run stability and disagreement," *The Review of Economic Studies*, forthcoming.
- COLEMAN, W. (1991): "Equilibrium in a production economy with an income tax," *Econometrica*, 59, 1091–1104.
- COUSOT, P. AND R. COUSOT (1979): "Constructive versions of Tarski's fixed point theorems," *Pacific Journal of Mathematics*, 82, 43–57.
- DATTA, M., K. REFFETT, AND L. WOŹNY (2018): "Comparing recursive equilibrium in economies with dynamic complementarities and indeterminacy," *Economic Theory*, 66, 593–626.
- DUGUNDJI, J. AND A. GRANAS (1982): Fixed Point Theory, Polish Scientific Publishers.
- ECHENIQUE, F. (2002): "Comparative statics by adaptive dynamics and the correspondence principle," *Econometrica*, 70, 833–844.
- (2003): "The equilibrium set of two-player games with complementarities is a sublattice," *Economic Theory*, 22, 903–905.
- (2005): "A short and constructive proof of Tarski's fixed-point theorem," International Journal of Game Theory, 33, 215–218.
- JACHYMSKI, J., L. GAJEK, AND P. POKAROWSKI (2000): "The Tarski-Kantorovitch prinicple and the theory of iterated function systems," *Bulletin of the Australian Mathematical Society*, 20, 247–261.

- KAMIHIGASHI, T. (2014): "Elementary results on solutions to the bellman equation of dynamic programming: existence, uniqueness, and convergence," *Economic Theory*, 56, 251–273.
- KIKUCHI, T., K. NISHIMURA, AND J. STACHURSKI (2018): "Span of control, transaction costs, and the structure of production chains," *Theoretical Economics*, 13, 729–760.
- KNASTER, B. AND A. TARSKI (1928): "Un théoremè sur les fonctions d'ensembles," Annales de la Societe Polonaise Mathematique, 6, 133–134.
- KUNIMOTO, T. AND T. YAMASHITA (2020): "Order on types based on monotone comparative statics," *Journal of Economic Theory*, 189, 105082.
- LI, H. AND J. STACHURSKI (2014): "Solving the income fluctuation problem with unbounded rewards," *Journal of Economic Dynamics and Control*, 45, 353–365.
- MILGROM, P. AND C. SHANNON (1994): "Monotone comparative statics," *Econometrica*, 62, 157–180.
- MIRMAN, L., O. MORAND, AND K. REFFETT (2008): "A qualitative approach to Markovian equilibrium in infinite horizon economies with capital," *Journal of Economic Theory*, 139, 75–98.
- OK, E. A. (2004): "Fixed set theory for closed correspondences with applications to self-similarity and games," *Nonlinear Analysis: Theory, Methods & Applications*, 56, 309–330.
- OLSZEWSKI, W. (2021a): "On convergence of sequences in complete lattices," Order, 38, 251–255.

(2021b): "On sequences of iterations of increasing and continuous mappings on complete lattices," *Games and Economic Behavior*, 126, 453–459.

- SABARWAL, T. (2023a): "General theory of equilibrium in models with complementarities," Tech. rep., University of Kansas, Department of Economics.
- (2023b): "Universal theory of equilibrium in models with complementarities," Tech. rep., University of Kansas, Department of Economics.
- SAMUELSON, P. A. (1947): Foundations of Economic Analysis, vol. 80 of Harvard Economic Studies, Harvard University Press, Cambridge.
- TARSKI, A. (1955): "A lattice-theoretical fixpoint theorem and its applications," Pacific Journal of Mathematics, 5, 285–309.
- TOPKIS, D. M. (1979): "Equilibrium points in nonzero-sum n-person submodular games," SIAM Journal of Control and Optimization, 17, 773–787.
- (1998): Supermodularity and Complementarity, Frontiers of economic research, Princeton University Press.
- VAN ZANDT, T. (2010): "Interim Bayesian Nash equilibrium on universal type spaces for supermodular games," *Journal of Economic Theory*, 145, 249–263.
- VEINOTT (1992): Lattice programming: qualitative optimization and equilibria, Technical

Report, Stanford.

- VIVES, X. (1990): "Nash equilibrium with strategic complementarites," Journal of Mathematical Economics, 19, 305–321.
- ZHOU, L. (1994a): "The set of Nash equilibria of a supermodular game is a complete lattice," *Games and Economic Behavior*, 7, 295–300.
 - (1994b): "The set of Nash equilibria of a supermodular game is a complete lattice," *Games and Economic Behavior*, 7, 295–300.