MICROECONOMICS 1

Master M1 QEM

PROBLEM SETS – Part A: Individual Decision Making

Consumer Theory

$L = 2$ is the number of commodities and $\mathbb{R}^2_+$ is the consumption set of the consumer.

Exercise 1 (Lexicographic preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

$x \succsim \overline{x} \iff$ " $x_1 > \overline{x}_1$ " or " $x_1 = \overline{x}_1$ and $x_2 \geq \overline{x}_2$ "

1. For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the upper contour set $U(\overline{x})$.
2. Show that for every $\overline{x} \in \mathbb{R}^2_+$, the indifference set $I(\overline{x})$ is a singleton.
3. Show that this preference relation is strictly monotone and strictly convex, but not continuous.

Exercise 2 (Linear preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \overline{x} \iff ax_1 + bx_2 \geq a\overline{x}_1 + b\overline{x}_2$$

with $a > 0$ and $b > 0$.

1. For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.
2. Show that this preference relation is continuous, convex, strictly monotone, but not strictly convex.

Exercise 3 (Leontief preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \overline{x} \iff \min\{x_1, x_2\} \geq \min\{\overline{x}_1, \overline{x}_2\}$$

1. For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.
2. Show that this preference relation is continuous, convex, monotone, but it is not strictly convex and it is not strictly monotone.

Exercise 4. Let $p = (p_1, p_2) \gg 0$ be a price system and let $w > 0$ be the wealth of the consumer. Using the definition of the demand of the consumer, determine **graphically** the demand of the consumer in the three following cases.

1. Lexicographic preferences.
2. Linear preferences.
3. Leontief preferences.

Exercise 5 (Cobb-Douglas utility function). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$,

$$u(x_1, x_2) = (x_1)^{\alpha}(x_2)^{1-\alpha} \text{ with } 0 < \alpha < 1$$

1. For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.

2. Determine the following properties of u: continuity, differentiability, (strictly) increasing, (strictly) (quasi-)concavity.

   Let $p = (p^1, p^2) \gg 0$ be a price system and $w > 0$ be the wealth of the consumer.

3. Show that if $x^* = (x^{*1}, x^{*2})$ belongs to the demand of this consumer, then $x^* \gg 0$.

4. Verify that the following utility function represents the Cobb-Douglas preferences on the interior of $\mathbb{R}^2_+$:
$$\widetilde{u}(x^1, x^2) = \alpha \ln x^1 + (1-\alpha) \ln x^2$$

5. Determine the following properties of $\widetilde{u}$: differentiability, (strictly) increasing, (strictly) (quasi-) concavity.

6. Determine the demand of this consumer.

Exercise 6. As usual, $x(p_1, p_2, \mathrm{w}) = (x_1(p_1, p_2, \mathrm{w}), x_2(p_1, p_2, \mathrm{w}))$ denotes the demand of the consumer. For every commodity $\ell = 1, 2$, the demand of commodity ℓ is given by

$$x_\ell(p_1, p_2, \mathrm{w}) = \frac{\mathrm{w}}{p_1 + p_2}$$

1. Prove that this demand is homogeneous of degree zero.

2. Prove that this demand satisfies Walras' Law.

3. State the Weak Axiom of Revealed Preferences (WARP) in the framework of the demand.

4. Prove that this demand satisfies WARP.

Exercise 7. Let L be the number of commodities. Let $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ be the consumption set. The consumer has strictly convex preferences which are represented by a utility function $u(x) = x_1 + \varphi(x_2, x_3, \ldots, x_L)$. We assume $p \gg 0$, and we normalize $p_1 = 1$.

1. Show that the demand for commodities $\{2, 3, \ldots, L\}$ must be independent of wealth. How does demand for commodity 1 react to changes in wealth w?

2. Using the previous result, define the indirect utility function as usual, i.e.
   $v(p, w) := u(x^*)$, where x^* belongs to the demand, given p and w. Show that $v(p, w)$ is linear in wealth: $v(p, w) = w + \psi(p)$ for some function $\psi : \mathbb{R}^L_{++} \to \mathbb{R}$.

3. Now let $L = 2$ and $\varphi(x_2) = \alpha \ln(x_2)$. Solve the UMP as a function of (p, w) (*Recall that we allow demand for commodity* 1 *to be negative*).

Production and Firm Behaviors

In both Exercises 8 and 9, the **basic properties** of the production set Y to be verified are the following ones:

- Possibility of inaction
- Closedness
- Impossibility of free production ("no free lunch")
- Free-disposal
- Irreversibility
- Convexity
- Increasing/decreasing/constant returns to scale.

Exercise 8. $L = 2$ is the number of commodities. A firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

1. Determine, both formally and graphically, the production set Y that corresponds to the production function f.

2. Determine if the production Y verifies the basic properties.

Now answer questions 1) and 2) for the two alternative production functions below:

- $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.

- $f(z) = \alpha(z)^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$.

Exercise 9. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0$$

1. Write the production set Y determined by the production function f.

2. Determine if the production Y verifies the basic properties.

Exercise 10. Let L be the finite number of commodities. A firm produces commodity L using the other $L-1$ commodities as inputs. $z := (z_1, ..., z_l, ..., z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}_+^{L-1} \longrightarrow \mathbb{R}_+$ is concave, then the transformation function defined by

$$t_f(y) := y_L - f(z)$$

is quasi-convex on the convex set $A = \{y = (-z, y_L) \in \mathbb{R}^L : z \geq 0 \text{ and } y_L \geq 0\}$.

Exercise 11. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

1. Write the profit maximization problem of this firm.

2. Consider the production set Y determined by the production function f. Using the shape of Y and the iso-profit lines, determine **graphically** the supply of this firm.

3. Determine the profit function of this firm.

Exercise 12. Let L be the finite number of commodities. Assume that the production set Y of the firm is represented by a transformation function $t: \mathbb{R}^L \to \mathbb{R}$, such that $Y = \{y \in \mathbb{R}^L : t(y) \leq 0\}$.

1. State the profit maximization problem (PMP) of the firm.
2. Let t be continuous and strictly quasi-convex. Show that if PMP has a solution for $p \gg 0$, then it must be unique.

Exercise 13. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity z as an input. The production function is given by $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.

1. Write the transformation function and the profit maximization problem (PMP) of this firm.
2. Show that if $\bar{y} = (\bar{y}_1, \bar{y}_2)$ belongs to the supply of the firm, then $\bar{y}_1 < 0$ and $\bar{y}_2 > 0$.
3. Consider the open and convex set $A = \{y = (-z, y_2) \in \mathbb{R}^2 : z > 0 \text{ and } y_2 > 0\}$. Write the first order conditions associated with (PMP) on the set A.
4. Compute the supply and the profit function of this firm.

Exercise 14. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity 1 as an input.

1. The production function is $f(z) = \alpha(1 - \exp(-kz))$ with $k > 0$, $\alpha > 0$ and $z \geq 0$.
   - Determine and draw the production set Y determined by the production function f.
   - For every level of output $\overline{y}_2 \geq 0$, determine and draw the following set
   $$Y(\overline{y}_2) := \{z \in \mathbb{R} : z \geq 0 \text{ and } f(z) \geq \overline{y}_2\}$$
   - Write the cost minimization problem of this firm.
   - Determine the demand of inputs and the cost function of the firm.
2. The production function is $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$, same questions.
3. The production function is $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$, same questions.

Exercise 15. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0$$

with $\alpha + \beta \leq 1$. Determine the demand of inputs and the cost function of the firm.

Exercise 16. Let L be the number of commodities. A firm produces commodity L using the other $L - 1$ commodities as inputs. $z := (z_1, ..., z_l, ..., z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}_+^{L-1} \longrightarrow \mathbb{R}_+$ is concave, then the cost function C is a convex function of the output level.

Exercise 17. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The cost function is given by

$$C(p^1, p^2, \bar{y}^3) = 2\left(\bar{y}^3\right)^2 \left(p^1\right)^{\frac{2}{3}} \left(p^2\right)^{\frac{1}{3}}$$

- Show that the cost function is homogeneous of degree one in (p^1, p^2).
- Verify that this cost function is a convex function of the output level.
- Compute the supply and the profit function of the firm.

Exercise 18. $L = 3$ is the number of commodities. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0$$

Using the demand of inputs and the cost function already determined in Exercise 15, determine the supply and the profit function of the firm [*Suggestion*: Distinguish the two cases: $\alpha + \beta < 1$ and $\alpha + \beta = 1$].

PROBLEM SETS – Part B: Equilibria and Optimality

Exercise 19. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are linear functions given by

$$u_1(x_{11}, x_{12}) = x_{11} + x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = ax_{21} + bx_{22}$$

$e_1 = (2, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

1. Draw the Edgeworth box associated with this economy and represent the feasible allocation (e_1, e_2).
2. Consider the point $(1, \frac{5}{2})$ and determine the individual consumptions associated with this point.
3. Write the definition of a competitive equilibrium for this specific pure exchange economy.
4. Take $a = 1$ and $b = 1$,
   - Represent the indifference curves of both consumers in the Edgeworth box.
   - Using the definition and the properties of a competitive equilibrium, determine **geometrically** the competitive equilibria of this pure exchange economy.
5. Take $a = 1$ and $b = 2$, same questions.

Exercise 20. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are Cobb-Douglas functions given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}$$

$e_1 = (1, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

1. Draw the Edgeworth box and represent the indifference curves of both consumers.
2. Remind the relevant market clearing condition associated with a competitive equilibrium.
3. Compute the competitive equilibrium $((p^*, 1), x_1^*, x_2^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^4_{++}$ (the price of commodity 2 has been normalized to 1), and represent $((p^*, 1), x_1^*, x_2^*)$ in the Edgeworth box.
4. Determine the redistribution of initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$ for which:

- both consumers have the same initial endowment of commodity 1, and
- the competitive equilibrium $((p^*, 1), x_1^*, x_2^*)$ is still a competitive equilibrium of the pure exchange economy with initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$.

Exercise 21. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The utility functions are of the Cobb-Douglas type, that is for all $i = 1, 2$

$$u_i(x_{i1}, x_{i2}) = (x_{i1})^{\alpha_i}(x_{i2})^{1-\alpha_i} \text{ with } \alpha_i \in]0, 1[$$

The initial endowments are $e_1 = (e_{11}, e_{12}) \gg 0$ and $e_2 = (e_{21}, e_{22}) \gg 0$.

1. Compute the aggregate excess demand function of this economy and show that it satisfies the *gross substitute property*, i.e. if the price of one commodity increases and the other one is kept fixed, then the aggregate excess demand of the other commodity strictly increases.

2. Show that if all consumers have the same initial endowments $e_1 = e_2 = e := (e_1, e_2)$, then the aggregate excess demand is the same as the one of a unique consumer having initial endowment $2e$ and a utility function given by $u(x_1, x_2) = (x_1)^{\alpha_1+\alpha_2}(x_2)^{2-\alpha_1-\alpha_2}$. Determine the equilibrium price as a function of e, α_1 and α_2.

3. Show that if all consumers have the same utility function, then the aggregate excess demand is the same as the one of a unique consumer with the same utility function and initial endowment $r := e_1 + e_2$. Determine the equilibrium price as a function of r and α_1.

Exercise 22. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are linear functions given by

$$u_1(x_{11}, x_{12}) = x_{11} + x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = ax_{21} + bx_{22}$$

$e_1 = (2, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

1. Write the definition of a Pareto optimal allocation for this specific pure exchange economy.
2. First, take $a = b = 1$. Using the definition of a Pareto optimal allocation, show **analytically** that the set of Pareto optimal allocations coincides with the set of feasible allocations.
3. From now on, take $a = 1$ and $b = 2$. Draw the Edgeworth box and represent the indifference curves of both consumers.
4. Using the Edgeworth box and the definition of a Pareto optimal allocation, determine **geometrically** the set of all Pareto optimal allocations.

Exercise 23. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{y = (y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } \alpha y_1 + y_2 \leq 0\}$$

with $\alpha > 0$. The two consumers have the same preferences represented by the utility function $u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$ for every $i = 1, 2$. The initial endowments are $e_1 = (1, 2)$ and $e_2 = (4, 1)$. The price of commodity 1 is normalized to 1, i.e. $p^1 = 1$.

1. Compute the demand of the consumers with respect to the price p^2 and the wealth $w > 0$.
2. Compute the supply and the profit function of the producer with respect to the price p^2 and the marginal productivity α.
3. The shares of the consumers on the profit of the firm have no influence on the competitive equilibria of this economy. Why so?
4. Compute the unique competitive equilibrium of this economy with respect to α.
5. Compute the utility level of the consumers with respect to the marginal productivity α. Show that the utility level of the second consumer is increasing. Show that the utility of the first consumer is constant, then decreasing and finally increasing.

Exercise 24. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{y = (y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_1 + y_2 \leq 0\}$$

The utility functions are given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}} + (x_{22})^{\frac{1}{2}}$$

and the aggregate initial endowment is $r = (2, 1)$.

One looks for all Pareto optimal allocations $(x_1^*, x_2^*, y^*) = ((x_{11}^*, x_{12}^*), (x_{21}^*, x_{22}^*), (y_1^*, y_2^*))$ of this economy with $y^* \neq 0$ and $(x_1^*, x_2^*) \gg 0$.

1. Remind the proposition on the characterization of Pareto optimal allocations in a differentiable framework.
2. Show that $\nabla u_1(x_{11}^*, x_{12}^*)$ and $\nabla u_2(x_{21}^*, x_{22}^*)$ are positively proportional to $(1, 1)$ and that $y^* = (-t, t)$ with $t > 0$.
3. Show that $x_{12}^* = 2x_{11}^*$ and $x_{21}^* = x_{22}^*$.
4. Show that all Pareto optimal allocations satisfying the required conditions are given by

$$((-1 + 2t, -2 + 4t), (3 - 3t, 3 - 3t), (-t, t)) \text{ with } t \in \left]\frac{1}{2}, 1\right[$$

Exercise 25. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The initial endowments are $e_1 = (1, 1)$ and $e_2 = (1, 1)$. The consumption set of both consumers is $\mathbb{R}^2_{++}$, the utility functions are given by

$$u_1(x_{11}, x_{12}) = \frac{1}{3} \ln x_{11} + \frac{2}{3} \ln x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = \frac{1}{4} \ln x_{21} + \frac{3}{4} \ln x_{22}$$

1. Determine the set of all Pareto optimal allocations.
2. For equity of treatment, the planner wishes to obtain a Pareto optimal allocation which guarantees the same consumption in commodity 1 for both consumers. Determine this specific Pareto optimal allocation $x^* = (x_1^*, x_2^*) \gg 0$.
3. In order decentralize this Pareto optimal allocation $x^* = (x_1^*, x_2^*)$, the planner has the possibility to implement some transfer between the initial endowments of commodity 1. Determine the transfer which leads to a competitive equilibrium satisfying the equity of treatment.