1. Consider a private ownership economy with $L = 2$ commodities, $m = 2$ consumers and $n = 2$ firms. Both firms produce the commodity 1 using the commodity 2 as an input. The production sets of the firms are defined below.

$$Y_1 = \{(y_{11}, y_{12}) \in \mathbb{R}^2 \colon y_{11} \leq -\frac{1}{2}y_{12} \text{ and } y_{12} \leq 0\} \text{ and } Y_2 = \{(y_{21}, y_{22}) \in \mathbb{R}^2 \colon y_{21} \leq \sqrt{-y_{22}} \text{ and } y_{22} \leq 0\}$$

The individual utility functions are given by

$$u_1(x_{11}, x_{12}) = x_{12}(x_{11})^2 \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{2}{3}}(x_{22})^{\frac{1}{3}}$$

The initial endowments are given by $e_1 = (2, 10)$ and $e_2 = (2, 6)$. Consumer 1 receives the whole profit of firm 1 and $\frac{1}{3}$ of the profit of firm 2 (consumer 2 receives the rest). The price of commodity 2 is normalized to 1, i.e. $p_2 = 1$.

(a) Write the definition of a competitive equilibrium for this specific private ownership economy.

(b) Compute the supplies and the optimal profits of firm 1 and firm 2.

(c) Using the optimal profits of the firms, compute the demands of both consumers.

(d) Determine the set of competitive equilibria.

2. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_2 \leq -y_1\}$$

The utility functions are given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}} + (x_{22})^{\frac{1}{2}}$$

and the aggregate initial endowment is $r = (2, 1)$.

One looks for all Pareto optimal allocations $(x_1^*, x_2^*, y^*) = ((x_{11}^*, x_{12}^*), (x_{21}^*, x_{22}^*), (y_1^*, y_2^*))$ of this economy with $y^* \neq 0$ and $(x_1^*, x_2^*) \gg 0$.

(a) Show that $y^* = (-t, t)$ with $t > 0$ and that $\nabla u_1(x_{11}^*, x_{12}^*)$ and $\nabla u_2(x_{21}^*, x_{22}^*)$ are positively proportional to $(1, 1)$.

(b) Show that $x_{12}^* = 2x_{11}^*$ and $x_{21}^* = x_{22}^*$.

(c) Show that all Pareto optimal allocations satisfying the required conditions are given by $((-1 + 2t, -2 + 4t), (3 - 3t, 3 - 3t), (-t, t))$ with $t \in]\frac{1}{2}, 1[$.

3. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_2 \leq -\alpha y_1\}$$

with $\alpha > 0$. The two consumers have the same preferences represented by the utility function $u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$ for every $i = 1, 2$. The initial endowments are $e_1 = (1, 2)$ and $e_2 = (4, 1)$. The price of commodity 1 is normalized to 1, i.e. $p^1 = 1$.

(a) Compute the demand of the consumers with respect to the price p^2 and the wealth $w > 0$.

(b) Compute the supply and the profit function of the producer with respect to the price p^2 and the marginal productivity α.

(c) The shares of the consumers on the profit of the firm have no influence on the competitive equilibria of this economy. Why so?

(d) Compute the unique competitive equilibrium of this economy with respect to α.

(e) Compute the utility level of the consumers with respect to the marginal productivity α. Show that the utility level of the second consumer is increasing. Show that the utility of the first consumer is constant, then decreasing and finally increasing.

(f) Can you explain the differences of the behavior of the utility levels of the consumers with respect to the marginal productivity?

4. In a certain economy there are three commodities, the time for leisure/for labor (commodity 1), the education (commodity 2) and the food (commodity 3). There is a single consumer with the utility function

$$u(x_2, x_3) = x_2 x_3$$

which depends only on the consumption of education x_2 and the consumption of food x_3, i.e., his consumption of leisure x_1 does not have any impact on his preferences. The consumer has an endowment of leisure equal to 1 and no endowment of education and food.

There are two firms. Firm 1 produces the education using the labor as the input according to the production function $f_1(y_{11}) = -\frac{1}{2}y_{11}$ with $y_{11} \leq 0$. Firm 2 produces the food using the labor as the input according to the production function $f_2(y_{21}) = 2\sqrt{-y_{21}}$ with $y_{21} \leq 0$.

(a) For every firm $j = 1, 2$, write the production set Y_j of firm j.

(b) Let $(\overline{x}, \overline{y}_1, \overline{y}_2) \in [0, 1] \times \mathbb{R}^2_+ \times Y_1 \times Y_2$ be a Pareto optimal allocation of this economy. Using the definition of a Pareto optimal allocation, prove that $\overline{x}_1 = 0$.

(c) Using question 2, first deduce the maximization problem under constraints which completely characterizes the set of Pareto optimal allocations. Second, using the Karush-Kuhn-Tucker conditions associated with that problem, determine the unique Pareto optimal allocation of this economy.

(d) From now on, $p = (1, p_2, p_3) \gg 0$ denotes a price system (i.e., the price of commodity 1 has been normalized to 1). We assume that all the agent are price-tacker and

- both firms are competitive,
- the consumer is the only owner of the firms.

Write the profit maximization problems of the firms. Write the utility maximization of the consumer and prove that his demand of leisure must be equal to zero.

(e) Consider the Pareto optimal allocation provided by question 3 and compute the supporting price p^* for which the Pareto optimal allocation is the equilibrium allocation of this economy.