Consumer Theory

1. Let $p = (p_1, p_2) \gg 0$ be a price system and $w > 0$ be the wealth of the consumer. Reconsider the Cobb-Douglas preferences from the previous problem set.
   (a) Show that if $x^* = (x_1^*, x_2^*)$ belongs to the demand of the consumer, then $x^* \gg 0$.
   (b) Verify that the following utility function represents the Cobb-Douglas preferences on the interior of $\mathbb{R}^2_+$:
   $$\widetilde{u}(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2$$
   (c) Determine the following properties of $\widetilde{u}$: is it differentiable, (strictly) increasing, (strictly) (quasi-) concave?
   (d) Determine the demand of the consumer.

2. Let $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ be the consumption set. The consumer has strictly convex preferences which are represented by a utility function $u(x) = x_1 + \varphi(x_2, x_3, \ldots, x_L)$. We assume $p \gg 0$, and we normalize $p_1 = 1$.
   (a) Try to show that the demand for commodities $\{2, 3, \ldots, L\}$ must be independent of wealth. How does demand for commodity 1 react to changes in wealth w?
   (b) Using your previous result, define the indirect utility function as usual, such that $v(p, w) = u(x^*)$, where x^* belongs to the demand, given p and w. Show that $v(p, w)$ is linear in wealth: $v(p, w) = w + \psi(p)$ for some function $\psi : \mathbb{R}^L_{++} \to \mathbb{R}$ (*You do not need to find* ψ).
   (c) Now let $L = 2$ and $\varphi(x_2) = \alpha \ln(x_2)$. Solve the UMP as a function of (p, w) (*Recall that we allow demand for commodity* 1 *to be negative*).

Producer Theory

3. A firm produces commodity 2 using commodity 1 as an input. The transformation function is $f(y_1) = \alpha(y_1)$ with $\alpha > 0$ and $y_1 \geq 0$.
   (a) Determine, both formally and graphically, the production set Y which corresponds to the production function f.
   (b) Determine the following basic properties of Y: possibility of inaction, closedness, impossibility of free production ("no free lunch"), free-disposal, irreversibility, convexity, increasing/decreasing/constant returns to scale.

4. Now answer all subquestion from the previous exercise for two alternative functions:
   (a) $f(y_1) = \alpha\sqrt{y_1}$ with $\alpha > 0$ and $y_1 \geq 0$.
   (b) $f(y_1) = \alpha(y_1)^2 + \beta y_1$ with $\alpha > 0,\ \beta > 0$ and $y_1 \geq 0$.

5. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by
   $$f(y_1, y_2) = (y_1)^\alpha (y_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ y_1 \geq 0 \text{ and } y_2 \geq 0$$
   (a) Determine the production set Y which corresponds to the production function f.
   (b) Determine the following basic properties of Y: possibility of inaction, closedness, impossibility of free production ("no free lunch"), free-disposal, irreversibility, convexity, increasing/decreasing/constant returns to scale.

6. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(y_1) = \alpha y_1$ with $\alpha > 0$ and $y_1 \geq 0$.
   (a) Write the profit maximization problem of this firm.
   (b) Consider the production set Y determined by the production function f. Using the shape of Y and the iso-profit lines, determine the supply of this firm.
   (c) Determine the profit function of this firm.

7. Let L be the finite number of commodities. Assume that the production set Y of the firm is represented by a transformation function $t : \mathbb{R}^L \to \mathbb{R}$, such that $Y = \{y \in \mathbb{R}^L : t(y) \leq 0\}$.
   (a) State the profit maximization problem (PMP) of the firm.
   (b) Let t be continuous and strictly quasi-convex. Show that if the PMP has a solution for $p \gg 0$, then it must be unique.