1. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are Cobb-Douglas functions given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}$$

$e_1 = (1, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

   (a) There exists a competitive equilibrium of this specific pure exchange economy. Why so?

   (b) Draw the Edgeworth box and represent the indifference curves of both consumers.

   (c) Compute the competitive equilibrium $((p^*, 1), x_1^*, x_2^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^4_{++}$ (the price of commodity 2 has been normalized to 1), and represent $((p^*, 1), x_1^*, x_2^*)$ in the Edgeworth box.

   (d) Determine the redistribution of initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$ for which:

   - both consumers have the same initial endowment of commodity 1, and
   - the competitive equilibrium $((p^*, 1), x_1^*, x_2^*)$ is still a competitive equilibrium of the pure exchange economy with initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$.

2. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The utility functions are of the Cobb-Douglas type, that is for all $i = 1, 2$

$$u_i(x_{i1}, x_{i2}) = (x_{i1})^{\alpha_i}(x_{i2})^{1-\alpha_i} \quad \text{with } \alpha_i \in]0, 1[$$

The initial endowments are $e_1 = (e_{11}, e_{12}) \gg 0$ and $e_2 = (e_{21}, e_{22}) \gg 0$.

   (a) Compute the aggregate excess demand function of this economy and show that it satisfies the *gross substitute property*, i.e. if the price of one commodity increases and the other one is kept fixed, then the aggregate excess demand of the other commodity strictly increases.

   (b) Show that if all consumers have the same initial endowments $e_1 = e_2 = e := (e_1, e_2)$, then the aggregate excess demand is the same as the one of a unique consumer having initial endowment $2e$ and a utility function given by $u(x_1, x_2) = (x_1)^{\alpha_1+\alpha_2}(x_2)^{2-\alpha_1-\alpha_2}$. Determine the equilibrium price as a function of e, α_1 and α_2.

   (c) Show that if all consumers have the same utility function, then the aggregate excess demand is the same as the one of a unique consumer with the same utility function and initial endowment $r := e_1 + e_2$. Determine the equilibrium price as a function of r and α_1.

3. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are linear functions given by

$$u_1(x_{11}, x_{12}) = x_{11} + x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = ax_{21} + bx_{22}$$

$e_1 = (2, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) Write the definition of a Pareto optimal allocation for this specific pure exchange economy.

(b) First, take $a = b = 1$. Using the definition of a Pareto optimal allocation, show **analytically** that the set of Pareto optimal allocations coincides with the set of feasible allocations.

(c) From now on, take $a = 1$ and $b = 2$. Draw the Edgeworth box and represent the indifference curves of both consumers.

(d) Using the Edgeworth box and the definition of a Pareto optimal allocation, determine **geometrically** the set of all Pareto optimal allocations.

4. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are Cobb-Douglas functions given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}$$

$e_1 = (1, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) Draw the Edgeworth box and represent the indifference curves of both consumers.

(b) Using the Edgeworth box and the definition of a Pareto optimal allocation, show that the allocations $(e_1 + e_2, (0, 0))$ and $((0, 0), e_1 + e_2)$ are the only Pareto optimal allocations on the boundary of the Edgeworth box.

(c) Using the first order conditions for Pareto optimality, compute all the Pareto optimal allocations $x^* = (x_1^*, x_2^*) \gg 0$.

(d) Represent the set of all Pareto optimal allocations in the Edgeworth box.

5. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. We assume that for every $i = 1, 2$, the preferences of consumer i are represented by the following utility function

$$u_i(x_{i1}, x_{i2}) = v_i(x_{i1}) + v_i(x_{i2})$$

where for every $i = 1, 2$, v_i is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying the following assumptions:

1. v_i is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$,
2. v_i is differentiable on $\mathbb{R}_{++}$ and the first derivative of v_i is strictly positive on $\mathbb{R}_{++}$.

Assume that the aggregate initial endowment $r = (r_1, r_2)$ is such that $r_1 = r_2 = \omega > 0$. Show that the Pareto optimal allocations are the allocations $(t_1 r, t_2 r)$ where $t_1 \geq 0$, $t_2 \geq 0$ and $t_1 + t_2 = 1$.

6. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The initial endowments are $e_1 = (1, 1)$ and $e_2 = (1, 1)$. The consumption set of both consumers is $\mathbb{R}^2_{++}$, the utility functions are given by

$$u_1(x_{11}, x_{12}) = \frac{1}{3} \ln x_{11} + \frac{2}{3} \ln x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = \frac{1}{4} \ln x_{21} + \frac{3}{4} \ln x_{22}$$

   (a) Find all Pareto optimal allocations.

   (b) For equity of treatment, the planner wishes to obtain a Pareto optimal allocation which guarantees the same consumption in commodity 1 for both consumers. Determine this specific Pareto optimal allocation $x^* = (x_1^*, x_2^*) \gg 0$.

   (c) In order decentralize this Pareto optimal allocation $x^* = (x_1^*, x_2^*)$, the planner has the possibility to implement some transfer between the initial endowments of commodity 1. Determine the transfer which leads to a competitive equilibrium satisfying the equity of treatment.