Midterm Exam

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PROBABILITY AND STATISTICS.
QEM Erasmus Mundus Master.
October 20th, 2015
Total time: 2:30h. Respond to all questions

First question (30 points)

Prove that:

1) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X, Y)$, where X and Y are random variables, and a and b are constants.

2) $\mathbb{E}^*[Y + Z|X] = \mathbb{E}^*[Y|X] + \mathbb{E}^*[Z|X]$ and that $\mathbb{E}[cY|X] = c\,\mathbb{E}[Y|X]$, where $\mathbb{E}[.]$ indicates expectation, and $\mathbb{E}^*[.]$ indicates the optimal linear predictor.

3) the optimal linear predictor is the best linear approximation to the conditional expectation function, that is $\mathbb{E}^*[Y|X] = \mathbb{E}^*[\mathbb{E}[Y|X]|X]$.

Second question (40 points)

Consider the joint probability mass function for (X, Y). We know that X takes values 1, 2, or 3 with probabilities $\frac{1}{3}$, $\frac{1}{3}$, and $\frac{1}{3}$ respectively. The variable Y is discrete, and its conditional distribution given X is Poisson, with parameter λX for $X = 1, 2, 3$ and $\lambda > 0$. [Recall that the pmf of a Poisson variable Z with parameter θ is $\frac{\theta^Z}{Z!}\exp(\theta)$, and that $\mathbb{E}[Z] = \text{Var}(Z) = \theta$.]

1) Derive the joint pmf and cdf of (X, Y) and show that they are well specified.

2) For $\lambda = 2$, compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X|Y = 1]$, $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Cov}(X, Y)$.

3) For $\lambda = 2$, compute the population R^2 for $\mathbb{E}[Y|X]$, and obtain the optimal linear predictor $\mathbb{E}^*[Y|X]$.

Third question (30 points)

After a soccer game that ends with a draw, the two teams are taken to a penalty shootout. The way the shootout works is as follows. Each team alternates a

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penalty quick until both teams shoot five each. If one of the teams scored more goals than the other, the team wins. If they scored the same number of goals, they both shoot an extra penalty. In the extra penalty, if one of them scores and the other does not, the one that scores wins. If either both of them score or none of them do, they shoot again another penalty quick each, and repeat the process until, in a round, one team scores and the other does not. Except when otherwise indicated, assume that if, after five extra rounds (i.e., the five initial shoots and five extra shoots for each team) no team wins, then the referee tosses a coin to decide which team wins. In each of the following cases, compute the probability that team 1 wins, the probability that after the initial five shoots for each team, the shootout still continues, and the probability that, conditional on entering to the extra shoots, team 1 wins.

1) Assume the probability of scoring a goal is $\frac{1}{2}$ for all players.

2) Assume that each team has five players, that better shooters shoot first, and that after all five shoot one, then they restart shooting in the same order, and that the probabilities of scoring for each of the five players are as follows: Team 1: $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{3}$; Team 2: all of them $\frac{1}{2}$.

3) All players have a probability $\frac{1}{2}$, but, instead of the tossing coin final stage, they keep shooting extra rounds until one of the teams wins.