Optimization: Tutorial 1 (Reminders)

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Exercise 4.3. Show that for $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$ the set $B(\mathbf{x}, r)$ is open; that is, show that an open ball is open.

Exercise 4.4. Show that for $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$ the closed ball $\overline{B(\mathbf{x}, r)}$ is closed.

Exercise 4.5. Show that any finite set of points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$ is closed.

Exercise 4.6. Show that in $\mathbb{R}^n$ no point $\mathbf{x}$ with $\|\mathbf{x}\| = 1$ is an interior point of $\overline{B(\mathbf{0}, 1)}$.

Exercise 4.7. Show that for any set S in $\mathbb{R}^n$ the set $\bar{S}$ is closed.

Exercise 4.8. Show that for any set S the closure $\bar{S}$ is equal to the intersection of all closed sets containing S.

Exercise 5.4. Let f be a real-valued function defined on a set S in $\mathbb{R}^n$. Show that if f is continuous at a point $\mathbf{x}_0$ in S and if $f(\mathbf{x}_0) < 0$, then there exists a $\delta > 0$ such that $f(\mathbf{x}) < 0$ for all $\mathbf{x}$ in $B(\mathbf{x}_0, \delta) \cap S$.

Exercise 5.5. Show that the real-valued function $f: \mathbb{R}^n \to \mathbb{R}^1$ defined by $f(\mathbf{x}) = \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ (i.e., show that the norm is a continuous function). *Hint:* Use the triangle inequality to show that, for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$, $|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,| \leqslant \|\mathbf{x} - \mathbf{y}\|$.

Exercise 6.3. Let A and B be two bounded sets of real numbers with $A \subseteq B$. Show that

$$\sup\{a : a \in A\} \leqslant \sup\{b : b \in B\},$$
$$\inf\{a : a \in A\} \geqslant \inf\{b : b \in B\}.$$

Exercise 6.4. Let A and B be two sets of real numbers.

(i) Show that if A and B are bounded above, then

$$\sup\{(a + b) : a \in A,\ b \in B\} = \sup\{a : a \in A\} + \sup\{b : b \in B\}.$$

(ii) Show that if A and B are bounded below, then

$$\inf\{(a + b) : a \in A,\ b \in B\} = \inf\{a : a \in A\} + \inf\{b : b \in B\}.$$

Exercise 2.1. Sketch the following sets in $\mathbb{R}^2$ and determine from your figure which sets are convex and which are not:

(a) $\{(x, y): x^2 + y^2 \leqslant 1\}$,
(b) $\{(x, y): 0 < x^2 + y^2 \leqslant 1\}$,
(c) $\{(x, y): y \geqslant x^2\}$,
(d) $\{(x, y): |x| + |y| \leqslant 1\}$, and
(e) $\{(x, y): y \geqslant 1/(1 + x^2)\}$.

Exercise 2.8. Consider the linear programming (LP) problem: Minimize $\langle \mathbf{c}, \mathbf{x} \rangle$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$. Let $S = \{\mathbf{x} : \mathbf{x}$ is a solution of the problem LP$\}$. Show that if S is not empty, then it is convex.

Exercise 2.9. Let $C \subseteq \mathbb{R}^n$. Show that C is convex if and only if $\lambda C + \mu C = (\lambda + \mu)C$ for all $\lambda \geqslant 0$, $\mu \geqslant 0$.

Exercise 2.10. A set C is said to be a *cone with vertex at the origin*, or simply *a cone*, if whenever $\mathbf{x} \in C$, all vectors $\lambda \mathbf{x}$, $\lambda \geqslant 0$, belong to C. If C is also convex, C is said to be a *convex cone*.

(a) Give an example of a cone that is not convex.
(b) Give an example of a cone that is convex.
(c) Let C be a nonempty set in $\mathbb{R}^n$. Show that C is a convex cone if and only if $\mathbf{x}_1$ and $\mathbf{x}_2 \in C$ implies that $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \in C$ for all $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$.

Exercise 2.11. Show that if C_1 and C_2 are convex cones, then so is $C_1 + C_2$ and that $C_1 + C_2 = \text{co}(C_1 \cup C_2)$.

Exercise 2.12. Show that if A is a bounded set in $\mathbb{R}^n$, then so is $\text{co}(A)$.

Exercise 1.1. Show that if f and g are convex functions defined on a convex set C, then for any $\lambda > 0$, $\mu > 0$, the function $\lambda f + \mu g$ is convex.

Exercise 1.2. Let $\{f_n\}$ be a sequence of convex functions defined on a convex set C. Show that if $f(\mathbf{x}) = \lim_{n \to \infty} f_n(\mathbf{x})$ exists for all $x \in C$, then f is convex.

Exercise 1.6. (a) Show that any norm v on $\mathbb{R}^n$ is convex on $\mathbb{R}^n$.

(b) Let S be a nonempty convex set, and let $\|\cdot\|$ denote the euclidean norm in $\mathbb{R}^n$. Let f be the distance function to S defined by

$$f(\mathbf{y}) = \inf\{\|\mathbf{y} - \mathbf{x}\| : \mathbf{x} \in S\}.$$

Show that f is convex on $\mathbb{R}^n$.

(c) Show that f is Lipschitz continuous on compact subsets of $\mathbb{R}^n$.