Probability

3. σ-algebra, independence, probability (finite case) and conditional probability, random variables

σ-algebra

Exercise 1 Let f be a function mapping Ω to another space E with a σ-algebra $\mathcal{E}$. Let

$$\mathcal{A} = \{A \subset \Omega, \exists B \in \mathcal{E}, A = f^{-1}(B)\}.$$

Show that $\mathcal{A}$ is a σ-algebra on Ω.

Exercise 2 Let Ω be a infinite sample space (countable or not), and let $\mathcal{A}$ be the family of all subsets of Ω which are finite or have a finite complement. Show that $\mathcal{A}$ is an algebra, but not a σ-algebra.

Exercise 3 Let $\mathcal{A}$ be a σ-algebra on a space Ω and $B \in \mathcal{A}$. Show that $\mathcal{F} = \{A \cap B;\ A \in \mathcal{A}\}$ is a σ-algebra of subsets of B. Is it still true when B is a subset of Ω that does not belong to $\mathcal{A}$?

Exercise 4 Let $(\mathcal{G}_\alpha)_{\alpha \in A}$ a family of σ-algebras defined on a space Ω. Show that $\cap_{\alpha \in A} \mathcal{G}_\alpha$ is also a σ-algebra.

Independence

Exercise 5 We flip a fair coin twice. Let us consider the three following events :
$A_1 = \{\text{head on first toss}\}$, $A_2 = \{\text{head on second toss}\}$ and $A_3 = \{\text{head on exactly one toss}\}$.
Show that A_1, A_2 and A_3 are pairwise independent but not independent.

Exercise 5' Show that if $A \cap B = \emptyset$ then A and B cannot be independent unless $P(A) = 0$ or $P(B) = 0$.

Probability

Exercise 6 Let Ω be a finite sample space, $\mathcal{A}$ be a σ-algebra on Ω, and P be a probability defined on Ω. Let A and B in $\mathcal{A}$ such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}.$$

Exercise 7 A box has r red and b black balls. A ball is chosen at random from the box (so that each ball is equally likely to be chosen), and then a second ball is drawn at random from the remaining balls in the box. Find the probabilities that

a) Both balls are red.

b) The first ball is red and the second is black.

Exercise 8 A urn contains N balls (balls have numbers $1, ..., N$). We choose n balls (with replacement) at random. Let X be the greatest number obtained and Y the smallest.

a) Find $P(X \leq x)$ for $x \in \{1, ..., N\}$ and deduce the distribution of X.

b) Find $P(Y \geq x)$ for $x \in \{1, ..., N\}$ and deduce the distribution of Y.

Exercise 9 A urn contains r red balls and b blue balls. A ball is chosen at random from the urn, its color is noted, and it is returned together with d more balls of the same color. This is repeated indefinitely. What is the probability that:

a) The second ball drawn is blue?

b) The first ball drawn is blue given that the second ball drawn is blue?

c) Let B_n denote the event that the n-th ball drawn is blue. Show that $P(B_n) = P(B_1)$ for all $n \geq 1$.

Conditional probability

Exercise 10 Donated blood is screened for some disease. Suppose that the test has 99% accuracy (meaning that $P(\text{test positive} \mid \text{you are ill}) = 0,99$), and that one in then thousand people in your age group are ill. The test has a 5% false positive rating, as well. Suppose the test screens you as positive. What is the probability you are ill? Is it 99%?

Exercise 11 Say we want to do a survey of undergraduate students at the colegio. It is known that 35% of the students are freshmen, 26% are sophomores, 22% are juniors and 17% are seniors. 75% of the freshmen say they like to eat in the cafeteria, versus 62% of the sophomores, 55% of the juniors and 40% of the seniors. If we randomly choose an undergraduate student who is eating in the cafeteria, what is the probability he/she is a senior?

Exercise 12 A factory produces light bulbs and has three different production shops. (A, B and C). A ensures 20% of the production, B 30% and C 50%. 5% of the light bulbs produced by A are faulty, 4% of the light bulbs produced by B are faulty and 1% of the light bulbs produced by C are faulty.

a) Find the probability that a light bulb produced by this factory is faulty.

b) We randomly choose a produced bulb and remark that it is is faulty. Find the probability that this bulb has been produced by B.

Exercise 13 I want to plant two types of plants in my garden, 30% type A and 70% type B. Suppose both type will either yield red or blue flowers. We know that $P(red \mid A) = 0.4$ and $P(red \mid B) = 0.3$.

a) What is the percentage of red flowers I will get ?

b) Suppose a red flower is picked randomly in my garden. What is the probability of the flower being type A ?

Exercise 14 Suppose A, B, C are independant events and $P(A \cap B) \neq \emptyset$. Show

$$P(C \mid A \cap B) = P(C).$$

Exercise 15 An insurance company insures an equal number of male and female drivers. In any given year, the probability that a mal driver has an accident involving a claim is α, independently of the other years. The analogous probability for females is β. Assume the insurance company selects a driver at random.

a) What is the probability the selected driver will make a claim this year ?

b) What is the probability the selected driver makes a claim in two consecutive years ?

c) Let A_1 and A_2 be the events that a randomly chosen driver makes a claim in each of the first and second years, respectively. Show that $P(A_2 \mid A_1) \geq P(A_1)$.

d) Find the probability that a claimant is female.

Random variables

Exercise 16 Let X and Y be uniform random variables on $\{0, 1, ..., n\}$. Suppose X and Y are independant. Find $P(X = Y)$ and $P(X \leq Y)$.

Exercise 17 Find the expectation and variance of a Bernoulli random variable.

Exercise 18 Let X and Y be two independent Bernouilli random variables with the same distributions.

a) Find the distribution of $X + Y$ and of $X - Y$.

b) Are $X + Y$ and $X - Y$ independent ?

Exercise 19 A fair coin is tossed 3 times. Let X equal 0 or 1 accordingly as a head or a tail occurs on the first toss, and let Y equal the total number of heads that occurs.

a) Find the distributions of X and Y.

b) Find the distribution of (X, Y).

c) Determine whether or not X and Y are independent.

d) Compute $cov(X, Y)$.

e) Find the distribution of $Z = X + Y$.

f) Compare $var(Z)$ and $var(X) + var(Y)$.

Exercise 20 We flip a fair coin n times. Find the distribution of the random variable X that is equal to the number of tails obtained. Find $E[X]$ and $var(X)$.

Exercise 21 Two players flip a fair coin n times. What is the probability that they obtain the same number of heads?

Exercise 22 Let X and Y be two independent binomial random variables, of parameter (n_1, p) and (n_2, p). Prove that $X + Y$ is a binomial random variable of parameter $(n_1 + n_2, p)$.

Exercise 23 Let X be a finite random variable. Prove that for every $a \in \mathbb{R}$,

$$E((X - a)^2) = Var(X) + (E(X) - a)^2.$$

Deduce from this result the infimum of the mapping $a \to E((X - a)^2)$.

Exercise 24 Let $b \in \mathbb{N}$ such that $b \geq 2$.

Let a box with one red ball and one green ball. One makes the following experience:

a) One first draws a ball.

b) If it is red, one stops the experience.

c) If it is green, one drops the green ball in the box, then one counts the number of green balls in the box (let n this number), and one add new green balls in the box in order to have $b.n$ green balls in the box, and one starts again in a).

Let X_b the the random variable defined as follows:

$X_b = 0$ if one neither stops in the experience above, and $X_b = n$ if one stops at draw n.

i) Compute

$$P(X_b = n)$$

for $n > 0$.

ii) Let u_n be the probability that after n draws, the experience is not finished. Prove that u_n converges when $n \to +\infty$.

Exercise 25 Let $n = 8.k$, where $k \in \mathbb{N}^*$. Consider the following game:

a) One asks you to choose two integers α and β such that

$$1 \leq \alpha < \beta < n$$

.

b) One draws three integers (independently, and each number with the same probability) X_1, X_2 and X_3 between 1 and n.

c) You win (in euros) X_3 if $X_3 > \beta$, X_2 if ($X_3 \leq \beta$ and $X_2 > \alpha$), and X_1 if ($X_3 \leq \beta$ and $X_2 \leq \alpha$).

Let $Z_{\alpha,\beta}$ the random variable equal to you winnings.

1) Let Y_α be the random variable equal to X_2 if $X_2 > \alpha$ and equal to X_1 if $X_2 \leq \alpha$.

i) Find the distribution of Y_α and its expectation.

ii) For what value of $\alpha \in \{1, ..., n-2\}$ is the expectation of Y_α maximal ? what is this maximal value ?

2) Prove that

$$P(Z_{\alpha,\beta} = k \mid X_3 \leq \beta) = P(Y_\alpha = k).$$

3) Deduce from 2) the expectation of $Z_{\alpha,\beta}$ (as a mapping of β, n and the expectation of Y_α.)

4) What are the best values of α and β you should choose ?