Probability
Continuous random variables

Exercise 1 We define f by $f(x) = k(4x - 2x^2)$ if $0 \leq x \leq 2$ and $f(x) = 0$ if $x \notin [0, 2]$.

1) Find k such that f is the density function of a random variable denoted by X.

2) Find $P(X > 1)$.

Exercise 2 Let X be a random variables distributed according a $\mathcal{E}(\lambda)$.

Show that $\forall x, y \geq 0$, $P(X > x + y | X > x) = P(X > x)$.

Exercise 3 Let us consider $X_1, ..., X_n$ i.i.d random variables distributed according a $\mathcal{U}([0, 1)$. Let $Y = Min(X_1, ..., X_n)$ and $Z = Max(X_1, ..., X_n)$.

1) Find the density of $V = \frac{1}{U}$.

2) Find the density function of Y and Z.

Exercise 4 Let X be a random variable with a distribution function F_X that is continuous and strictly increasing.

1) Show that $F_X(X) \hookrightarrow \mathcal{U}([0, 1)$.

2) If $U \hookrightarrow \mathcal{U}([0, 1)$, prove that the distribution function of $F_X^{-1}(U)$ is F_X.

Exercise 5 We consider the following integral

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha - 1} dy.$$

1) If $\alpha > 0$, prove that $\Gamma(\alpha)$ is well defined.

2) Show that $\Gamma(1) = 1$.

3) Prove that $\forall \alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ and deduce that $\forall n \in \mathbb{N}^*$, $\Gamma(n) = (n - 1)!$.

4) For $\lambda > 0$ and $\alpha > 0$ we define

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} I_{]0,+\infty[}(x).$$

Prove that f is a density function. If a random variable X has the density f we say that X follows a $\Gamma(\alpha, \lambda)$.

5) Show that $\Gamma(1, \lambda) = \mathcal{E}(\lambda)$.

6) If $X \hookrightarrow \mathcal{N}(0,1)$ prove that $X^2 \hookrightarrow \Gamma(2, 1)$.

7) Show that $E[X] = \frac{\alpha}{\lambda}$ and $Var(X) = \frac{\alpha}{\lambda^2}$.

Exercise 6 When $Z \hookrightarrow \mathcal{E}(\lambda)$ show that $\forall n \in \mathbb{N}$, $E[Z^n] = \frac{n!}{\lambda^n}$.

Exercise 7 Let (X, Y) be a pair of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and with values in $\mathbb{N} \times \mathbb{R}$ such that $\forall k \in \mathbb{N}$, $\forall B \in Bor(\mathbb{R})$,

$$P(X = k \cap Y \in B) = \int_{B \cap]0,+\infty[} e^{-y} \frac{y^k \theta^p}{k!(p-1)!} y^{p-1} e^{-\theta y} dy$$

with $p \in \mathbb{N}^*$ and $\theta > 0$.

Prove that $P(X = k) = \frac{\theta^p}{(1+\theta)^{k+1}} C_{k+p-1}^k$.