Final Exam (1 hour and 45 minutes)

Course Questions (20 minutes). Consider the following functions $f : x \in C \subseteq \mathbb{R}^n \to f(x) \in \mathbb{R}$, $g_j : x \in C \subseteq \mathbb{R}^n \to g_j(x) \in \mathbb{R}$ with $j = 1, ..., m$, and the maximization problem (P) given below.

$$(P) \qquad \max_{x \in C} \quad f(x)$$
$$\text{subject to} \quad \begin{cases} g_1(x) \geq 0 \\ \dots \\ g_j(x) \geq 0 \\ \dots \\ g_m(x) \geq 0 \end{cases}$$

1. Write the Karush–Kuhn–Tucker (KKT) conditions associated with the problem (P).

2. Let $x^* \in C$. What does mean that the constraint j is binding at x^* ? (Give the definition).

3. State the theorem for which KKT conditions are necessary conditions to solve problem (P).

4. State the theorem for which KKT conditions are sufficient conditions to solve problem (P).

Exercise 1 (15 minutes). State the proposition concerning the second order characterization of a strictly quasi-concave function. Use this proposition to prove that the following function

$$f(x_1, x_2) = 3x_1x_2 - (x_2)^3$$

is strictly quasi-concave on $C = \mathbb{R}^2_{++}$.

Exercise 2 (45 minutes). Consider the maximization problem (H) given below.

$$(H) \qquad \max_{(x_1,x_2) \in \mathbb{R}^2} \quad 3x_1x_2 - (x_2)^3$$
$$\text{subject to} \quad \begin{cases} x_1 > 0, \ x_2 > 0 \\ x_1 - 2x_2 \leq 5 \\ 2x_1 + 5x_2 \geq 20 \end{cases}$$

1. Draw the set of feasible points for problem (H).

2. Weierstrass' Theorem does not apply. Why so ? (Justify your answer).

3. Write the system determined by the KKT conditions associated with problem (H) and solve this system.

4. Discuss whether KKT conditions are necessary/sufficient in order to solve problem (H).

5. Determine $Sol(H)$.

Exercise 3 (20 minutes). Prove the theorem given in Course Question 3.