$L = 2$ is the number of commodities and $\mathbb{R}^2_+$ is the consumption set of the consumer.

1. (Linear preferences) For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \overline{x} \iff ax_1 + bx_2 \geq a\overline{x}_1 + b\overline{x}_2$$

   with $a > 0$ and $b > 0$. For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.

2. (Leontief preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

$$x \succsim \overline{x} \iff \min\{x_1, x_2\} \geq \min\{\overline{x}_1, \overline{x}_2\}$$

   For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.

3. (Lexicographic preferences). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$ and $\overline{x} = (\overline{x}_1, \overline{x}_2) \in \mathbb{R}^2_+$,

   $x \succsim \overline{x} \iff$ " $x_1 > \overline{x}_1$ " or " $x_1 = \overline{x}_1$ and $x_2 \geq \overline{x}_2$ "

   (a) For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the upper contour set $U(\overline{x})$.

   (b) Show that for every $\overline{x} \in \mathbb{R}^2_+$, the indifference set $I(\overline{x})$ is a singleton.

4. Reconsider linear preferences with $a > 0$ and $b > 0$. Show that this preference relation is continuous, convex, strictly monotone, but not strictly convex.

5. Reconsider Leontief preferences. Show that this preference relation is continuous, convex, monotone, but it is not strictly convex and it is not strictly monotone.

6. Reconsider Lexicographic preferences. Show that this preference relation is strictly monotone and strictly convex, but not continuous.

7. Let $p = (p_1, p_2) \gg 0$ be a price system and let $w > 0$ be the wealth of the consumer. Determine **graphically** the demand of the consumer for the three following cases.

   (a) Linear preferences.

   (b) Leontief preferences.

   (c) Lexicographic preferences.

8. (Cobb-Douglas utility function). For all $x = (x_1, x_2) \in \mathbb{R}^2_+$,

$$u(x_1, x_2) = (x_1)^\alpha (x_2)^{1-\alpha} \text{ with } 0 < \alpha < 1$$

   (a) For every $\overline{x} \in \mathbb{R}^2_+$, determine and draw the indifference curve $I(\overline{x})$ and the upper contour set $U(\overline{x})$.

   (b) Determine the following properties of u: continuity, differentiability, (strictly) increasing, (strictly) (quasi-)concavity.

Consumer Theory

1. Let $p = (p_1, p_2) \gg 0$ be a price system and $w > 0$ be the wealth of the consumer. Reconsider the Cobb-Douglas preferences from the previous problem set.
   (a) Show that if $x^* = (x_1^*, x_2^*)$ belongs to the demand of the consumer, then $x^* \gg 0$.
   (b) Verify that the following utility function represents the Cobb-Douglas preferences on the interior of $\mathbb{R}^2_+$:
   $$\widetilde{u}(x_1, x_2) = \alpha \ln x_1 + (1-\alpha) \ln x_2$$
   (c) Determine the following properties of $\widetilde{u}$: is it differentiable, (strictly) increasing, (strictly) (quasi-) concave?
   (d) Determine the demand of the consumer.

2. Let $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ be the consumption set. The consumer has strictly convex preferences which are represented by a utility function $u(x) = x_1 + \varphi(x_2, x_3, \ldots, x_L)$. We assume $p \gg 0$, and we normalize $p_1 = 1$.
   (a) Try to show that the demand for commodities $\{2, 3, \ldots, L\}$ must be independent of wealth. How does demand for commodity 1 react to changes in wealth w?
   (b) Using your previous result, define the indirect utility function as usual, such that $v(p, w) = u(x^*)$, where x^* belongs to the demand, given p and w. Show that $v(p, w)$ is linear in wealth: $v(p, w) = w + \psi(p)$ for some function $\psi : \mathbb{R}^L_{++} \to \mathbb{R}$ (*You do not need to find* ψ).
   (c) Now let $L = 2$ and $\varphi(x_2) = \alpha \ln(x_2)$. Solve the UMP as a function of (p, w) (*Recall that we allow demand for commodity* 1 *to be negative*).

Producer Theory

3. A firm produces commodity 2 using commodity 1 as an input. The production function is $f(y_1) = \alpha(y_1)$ with $\alpha > 0$ and $y_1 \geq 0$.
   (a) Determine, both formally and graphically, the production set Y which corresponds to the production function f.
   (b) Determine the following basic properties of Y: possibility of inaction, closedness, impossibility of free production ("no free lunch"), free-disposal, irreversibility, convexity, increasing/decreasing/constant returns to scale.

4. Now answer all subquestion from the previous exercise for two alternative functions:
   (a) $f(y_1) = \alpha\sqrt{y_1}$ with $\alpha > 0$ and $y_1 \geq 0$.
   (b) $f(y_1) = \alpha(y_1)^2 + \beta y_1$ with $\alpha > 0$, $\beta > 0$ and $y_1 \geq 0$.

5. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by
   $$f(y_1, y_2) = (y_1)^\alpha (y_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ y_1 \geq 0 \text{ and } y_2 \geq 0$$
   (a) Determine the production set Y which corresponds to the production function f.
   (b) Determine the following basic properties of Y: possibility of inaction, closedness, impossibility of free production ("no free lunch"), free-disposal, irreversibility, convexity, increasing/decreasing/constant returns to scale.

6. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(y_1) = \alpha y_1$ with $\alpha > 0$ and $y_1 \geq 0$.
   (a) Write the profit maximization problem of this firm.
   (b) Consider the production set Y determined by the production function f. Using the shape of Y and the iso-profit lines, determine the supply of this firm.
   (c) Determine the profit function of this firm.
7. Let L be the finite number of commodities. Assume that the production set Y of the firm is represented by a transformation function $t : \mathbb{R}^L \to \mathbb{R}$, such that $Y = \{y \in \mathbb{R}^L : t(y) \leq 0\}$.
   (a) State the profit maximization problem (PMP) of the firm.
   (b) Let t be continuous and strictly quasi-convex. Show that if the PMP has a solution for $p \gg 0$, then it must be unique.

1. $L = 2$ is the number of commodities. The firm produces commodity y_2 using commodity z as an input. The production function is given by $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.
   (a) Write the transformation function and the profit maximization problem (PMP) of this firm, where $y_1 = -z$ and $y_2 = f(z)$.
   (b) Show that if $\bar{y} = (y_1, y_2)$ belongs to the supply of the firm, then $y_1 < 0$ and $y_2 > 0$.
   (c) Consider the open and convex set $A = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 < 0 \text{ and } y_2 > 0\}$. Write the first order conditions associated with (PMP) on the set A, and determine if these conditions are necessary and/or sufficient to solve (PMP) on the set A.
   (d) Compute the supply and the profit function of the firm.

2. Let L be the finite number of commodities. A firm produces commodity L using the other $L - 1$ commodities as inputs. $z := (z_1, ..., z_l, ..., z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}_+^{L-1} \longrightarrow \mathbb{R}_+$ is differentiable and concave on the interior of $\mathbb{R}_+^{L-1}$, then the transformation function defined by

$$t_f(y) := y_L - f(z)$$

   is differentiable and quasi-convex on the open and convex set $A = \{y = (-z, y_L) \in \mathbb{R}^L \colon z \gg 0 \text{ and } y_L > 0\}$ [*Suggestion*: Use the first order characterization of concave and quasi-convex functions].

3. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta$$

   with $z_1 \geq 0, z_2 \geq 0, \alpha > 0,\ \beta > 0, \alpha + \beta \leq 1$.
   (a) Write the transformation function and the profit maximization problem (PMP) of this firm.
   (b) Consider the open and convex set $A = \{y = (-z_1, -z_2, y_3) \in \mathbb{R}^3 \colon z_1 > 0, z_2 > 0, \text{ and } y_3 > 0\}$. Write the first order conditions associated with (PMP) on the set A, and determine if these conditions are necessary and/or sufficient to solve (PMP) on the set A.
   (c) Compute the supply and the profit function of the firm [*Suggestion*: Distinguish the two cases $\alpha + \beta < 1$ and $\alpha + \beta = 1$].

4. Using the definition of the profit function π, prove that π is a convex function.

5. $L = 2$ is the number of commodities. The firm produces commodity 2 using commodity 1 as an input.
   (a) The production function is $f(z) = \alpha(1 - \exp(-kz))$ with $k > 0$, $\alpha > 0$ and $z \geq 0$.
      - Determine and draw the production set Y determined by the production function f.
      - For every level of output $\overline{y}_2 \geq 0$, determine and draw the following set

$$Y(\overline{y}_2) := \{z \in \mathbb{R} \colon z \geq 0 \text{ and } f(z) \geq \overline{y}_2\}$$

      - Write the cost minimization problem of this firm.
      - Determine the demand of inputs and the cost function of the firm.
   (b) The production function is $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$, same questions.
   (c) The production function is $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0,\ \beta > 0$ and $z \geq 0$, same questions.

6. $L = 3$ is the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0$$

   with $\alpha + \beta \leq 1$. Determine the demand of inputs and the cost function of the firm [*Suggestion*: Distinguish the two cases $\alpha + \beta < 1$ and $\alpha + \beta = 1$].

Cost Minimization

1. $L = 3$ is the number of commodities. The firm produces y using commodities 1 and 2 as inputs. The cost function is given by

$$C(w_1, w_2, y) = 2(y)^2\,(w_1)^{\frac{2}{3}}\,(w_2)^{\frac{1}{3}}$$

Compute the supply and the profit function of the firm.

2. Let L be the number of commodities. A firm produces commodity y using the other $L-1$ commodities as inputs. $z := (z_1, ..., z_l, ..., z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}^{L-1} \longrightarrow \mathbb{R}_+$ is concave, then the the cost function $C(w, y)$ is convex with respect to the output level $y \in \mathbb{R}_+$.

3. $L = 3$ is the number of commodities. The production function is given by

$$f(z_1, z_2) = (z_1)^{\alpha}(z_2)^{\beta} \text{ with } \alpha > 0,\ \beta > 0,\ z_1 \geq 0 \text{ and } z_2 \geq 0.$$

Use the demand of inputs and the cost function already determined in a previous exercise, determine the supply of the firm.

Equilibria and Optimality

4. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are linear functions given by

$$u_1(x_{11}, x_{12}) = x_{11} + x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = ax_{21} + bx_{22}$$

$e_1 = (2, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) Draw the Edgeworth box associated with this economy and represent the feasible allocation (e_1, e_2).

(b) Consider the point $(1, \frac{5}{2})$ and determine the individual consumptions associated with this point.

(c) Write the definition of a competitive equilibrium for this specific pure exchange economy.

(d) Take $a = 1$ and $b = 1$,

   1. Represent the indifference curves of both consumers in the Edgeworth box.
   2. Using the definition and the properties of competitive equilibria, determine **geometrically** the competitive equilibria of this pure exchange economy.

(e) Take $a = 1$ and $b = 2$, same questions.

1. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are Cobb-Douglas functions given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}$$

$e_1 = (1, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) There exists a competitive equilibrium of this specific pure exchange economy. Why so?

(b) Draw the Edgeworth box and represent the indifference curves of both consumers.

(c) Compute the competitive equilibrium $((p^*, 1), x_1^*, x_2^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^4_{++}$ (the price of commodity 2 has been normalized to 1), and represent $((p^*, 1), x_1^*, x_2^*)$ in the Edgeworth box.

(d) Determine the redistribution of initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$ for which:

- both consumers have the same initial endowment of commodity 1, and
- the competitive equilibrium $((p^*, 1), x_1^*, x_2^*)$ is still a competitive equilibrium of the pure exchange economy with initial endowments $(\widetilde{e}_1, \widetilde{e}_2)$.

2. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The utility functions are of the Cobb-Douglas type, that is for all $i = 1, 2$

$$u_i(x_{i1}, x_{i2}) = (x_{i1})^{\alpha_i}(x_{i2})^{1-\alpha_i} \text{ with } \alpha_i \in]0, 1[$$

The initial endowments are $e_1 = (e_{11}, e_{12}) \gg 0$ and $e_2 = (e_{21}, e_{22}) \gg 0$.

(a) Compute the aggregate excess demand function of this economy and show that it satisfies the *gross substitute property*, i.e. if the price of one commodity increases and the other one is kept fixed, then the aggregate excess demand of the other commodity strictly increases.

(b) Show that if all consumers have the same initial endowments $e_1 = e_2 = e := (e_1, e_2)$, then the aggregate excess demand is the same as the one of a unique consumer having initial endowment $2e$ and a utility function given by $u(x_1, x_2) = (x_1)^{\alpha_1+\alpha_2}(x_2)^{2-\alpha_1-\alpha_2}$. Determine the equilibrium price as a function of e, α_1 and α_2.

(c) Show that if all consumers have the same utility function, then the aggregate excess demand is the same as the one of a unique consumer with the same utility function and initial endowment $r := e_1 + e_2$. Determine the equilibrium price as a function of r and α_1.

3. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are linear functions given by

$$u_1(x_{11}, x_{12}) = x_{11} + x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = ax_{21} + bx_{22}$$

$e_1 = (2, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) Write the definition of a Pareto optimal allocation for this specific pure exchange economy.

(b) First, take $a = b = 1$. Using the definition of a Pareto optimal allocation, show **analytically** that the set of Pareto optimal allocations coincides with the set of feasible allocations.

(c) From now on, take $a = 1$ and $b = 2$. Draw the Edgeworth box and represent the indifference curves of both consumers.

(d) Using the Edgeworth box and the definition of a Pareto optimal allocation, determine **geometrically** the set of all Pareto optimal allocations.

4. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The individual utility functions are Cobb-Douglas functions given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}}(x_{22})^{\frac{1}{2}}$$

$e_1 = (1, 2)$ and $e_2 = (2, 1)$ are the initial endowments.

(a) Draw the Edgeworth box and represent the indifference curves of both consumers.

(b) Using the Edgeworth box and the definition of a Pareto optimal allocation, show that the allocations $(e_1 + e_2, (0, 0))$ and $((0, 0), e_1 + e_2)$ are the only Pareto optimal allocations on the boundary of the Edgeworth box.

(c) Using the first order conditions for Pareto optimality, compute all the Pareto optimal allocations $x^* = (x_1^*, x_2^*) \gg 0$.

(d) Represent the set of all Pareto optimal allocations in the Edgeworth box.

5. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. We assume that for every $i = 1, 2$, the preferences of consumer i are represented by the following utility function

$$u_i(x_{i1}, x_{i2}) = v_i(x_{i1}) + v_i(x_{i2})$$

where for every $i = 1, 2$, v_i is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying the following assumptions:

1. v_i is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$,
2. v_i is differentiable on $\mathbb{R}_{++}$ and the first derivative of v_i is strictly positive on $\mathbb{R}_{++}$.

Assume that the aggregate initial endowment $r = (r_1, r_2)$ is such that $r_1 = r_2 = \omega > 0$. Show that the Pareto optimal allocations are the allocations $(t_1 r, t_2 r)$ where $t_1 \geq 0$, $t_2 \geq 0$ and $t_1 + t_2 = 1$.

6. Consider a pure exchange economy with $L = 2$ commodities and $m = 2$ consumers. The initial endowments are $e_1 = (1, 1)$ and $e_2 = (1, 1)$. The consumption set of both consumers is $\mathbb{R}^2_{++}$, the utility functions are given by

$$u_1(x_{11}, x_{12}) = \frac{1}{3} \ln x_{11} + \frac{2}{3} \ln x_{12} \quad \text{and} \quad u_2(x_{21}, x_{22}) = \frac{1}{4} \ln x_{21} + \frac{3}{4} \ln x_{22}$$

   (a) Find all Pareto optimal allocations.

   (b) For equity of treatment, the planner wishes to obtain a Pareto optimal allocation which guarantees the same consumption in commodity 1 for both consumers. Determine this specific Pareto optimal allocation $x^* = (x_1^*, x_2^*) \gg 0$.

   (c) In order decentralize this Pareto optimal allocation $x^* = (x_1^*, x_2^*)$, the planner has the possibility to implement some transfer between the initial endowments of commodity 1. Determine the transfer which leads to a competitive equilibrium satisfying the equity of treatment.

1. Consider a private ownership economy with $L = 2$ commodities, $m = 2$ consumers and $n = 2$ firms. Both firms produce the commodity 1 using the commodity 2 as an input. The production sets of the firms are defined below.

$$Y_1 = \{(y_{11}, y_{12}) \in \mathbb{R}^2 \colon y_{11} \leq -\frac{1}{2}y_{12} \text{ and } y_{12} \leq 0\} \text{ and } Y_2 = \{(y_{21}, y_{22}) \in \mathbb{R}^2 \colon y_{21} \leq \sqrt{-y_{22}} \text{ and } y_{22} \leq 0\}$$

The individual utility functions are given by

$$u_1(x_{11}, x_{12}) = x_{12}(x_{11})^2 \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{2}{3}}(x_{22})^{\frac{1}{3}}$$

The initial endowments are given by $e_1 = (2, 10)$ and $e_2 = (2, 6)$. Consumer 1 receives the whole profit of firm 1 and $\frac{1}{3}$ of the profit of firm 2 (consumer 2 receives the rest). The price of commodity 2 is normalized to 1, i.e. $p_2 = 1$.

(a) Write the definition of a competitive equilibrium for this specific private ownership economy.

(b) Compute the supplies and the optimal profits of firm 1 and firm 2.

(c) Using the optimal profits of the firms, compute the demands of both consumers.

(d) Determine the set of competitive equilibria.

2. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_2 \leq -y_1\}$$

The utility functions are given by

$$u_1(x_{11}, x_{12}) = (x_{11})^{\frac{1}{3}}(x_{12})^{\frac{2}{3}} \quad \text{and} \quad u_2(x_{21}, x_{22}) = (x_{21})^{\frac{1}{2}} + (x_{22})^{\frac{1}{2}}$$

and the aggregate initial endowment is $r = (2, 1)$.

One looks for all Pareto optimal allocations $(x_1^*, x_2^*, y^*) = ((x_{11}^*, x_{12}^*), (x_{21}^*, x_{22}^*), (y_1^*, y_2^*))$ of this economy with $y^* \neq 0$ and $(x_1^*, x_2^*) \gg 0$.

(a) Show that $y^* = (-t, t)$ with $t > 0$ and that $\nabla u_1(x_{11}^*, x_{12}^*)$ and $\nabla u_2(x_{21}^*, x_{22}^*)$ are positively proportional to $(1, 1)$.

(b) Show that $x_{12}^* = 2x_{11}^*$ and $x_{21}^* = x_{22}^*$.

(c) Show that all Pareto optimal allocations satisfying the required conditions are given by $((-1 + 2t, -2 + 4t), (3 - 3t, 3 - 3t), (-t, t))$ with $t \in]\frac{1}{2}, 1[$.

3. Consider a production economy with $L = 2$ commodities, $m = 2$ consumers and one firm. The firm produces the commodity 2 using the commodity 1 as an input with constant returns to scale. The production set of the firm is given by

$$Y = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 \leq 0 \text{ and } y_2 \leq -\alpha y_1\}$$

with $\alpha > 0$. The two consumers have the same preferences represented by the utility function $u_i(x_{i1}, x_{i2}) = x_{i1}x_{i2}$ for every $i = 1, 2$. The initial endowments are $e_1 = (1, 2)$ and $e_2 = (4, 1)$. The price of commodity 1 is normalized to 1, i.e. $p^1 = 1$.

   (a) Compute the demand of the consumers with respect to the price p^2 and the wealth $w > 0$.

   (b) Compute the supply and the profit function of the producer with respect to the price p^2 and the marginal productivity α.

   (c) The shares of the consumers on the profit of the firm have no influence on the competitive equilibria of this economy. Why so?

   (d) Compute the unique competitive equilibrium of this economy with respect to α.

   (e) Compute the utility level of the consumers with respect to the marginal productivity α. Show that the utility level of the second consumer is increasing. Show that the utility of the first consumer is constant, then decreasing and finally increasing.

   (f) Can you explain the differences of the behavior of the utility levels of the consumers with respect to the marginal productivity?

4. Consider three commodities, the time for leisure/for labor (commodity 1), the education (commodity 2) and the food (commodity 3). There is a single consumer with the utility function

$$u(x_2, x_3) = x_2 x_3$$

which depends only on the consumption of education x_2 and the consumption of food x_3, i.e., his consumption of leisure x_1 does not have any impact on his preferences. The consumer has an endowment of leisure equal to 1 and no endowment of education and food.

   There are two firms. For every firm $j = 1, 2$, f_j denotes the production function of firm j. As usual, in the framework of the production function, the inputs are measured as positive quantities. Firm 1 produces only education using labor as the input according to the production function $f_1(z_{11}) = \frac{1}{2}z_{11}$ with $z_{11} \geq 0$. Firm 2 produces only food using the labor as the input according to the production function $f_2(z_{21}) = 2\sqrt{z_{21}}$ with $z_{21} \geq 0$.

   (a) As usual, let $y_j = (y_{j1}, y_{j2}, y_{j3}) \in Y_j$ denote a feasible production choice of firm j. For both firms $j = 1, 2$, write the above technology in terms of the production set $Y_j \subseteq \mathbb{R}^3$.

   (b) Let $(\overline{x}, \overline{y}_1, \overline{y}_2) \in [0, 1] \times \mathbb{R}^2_+ \times Y_1 \times Y_2$ be a Pareto optimal allocation of this economy. Using the definition of a Pareto optimal allocation, prove that $\overline{x}_1 = 0$.

(c) Using question 2, first deduce the maximization problem under constraints which completely characterizes the set of Pareto optimal allocations. Second, using the Karush-Kuhn-Tucker conditions associated with that problem, determine the unique Pareto optimal allocation of this economy.

(d) From now on, $p = (1, p_2, p_3) \gg 0$ denotes a price system (i.e., the price of commodity 1 has been normalized to 1). We assume that all the agent are price-tacker and

- both firms are competitive,
- the consumer is the only owner of the firms.

Write the profit maximization problems of the firms. Write the utility maximization of the consumer and prove that his demand of leisure must be equal to zero.

(e) Consider the Pareto optimal allocation provided by question 3 and compute the supporting price p^* for which the Pareto optimal allocation is the equilibrium allocation of this economy.