Optimization. A first course on mathematics for economists
Problem set 4: Classical programming

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4.1 Let $f(x_1, x_2) = 2x_1^2 + x_2^2$. Solve the following problem:

$$\min_{x_1, x_2} 2x_1^2 + x_2^2 \text{ s.t.}$$
$$x_1 + x_2 = 1$$

Give a geometric interpretation to the solution.

Solution: *The Lagrangean function is*

$$L(x_1, x_2, \lambda) = 2x_1^2 + x_2^2 + \lambda(1 - x_1 - x_2)$$

The first-order conditions (FOCs) are

$$\frac{\partial L}{\partial x_1} = 4x_1 - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 1 - x_1 - x_2 = 0$$

From the first two equations we obtain, $2x_1 = x_2$. *Substituting in the third equation gives the solution:*

$$x_1^* = 1/3, \quad x_2^* = 2/3, \quad \lambda^* = 4/3, \quad f(x_1^*, x_2^*) = 2/3.$$

To assess that the solution is actually minimizing the objective function f*, we look at the second order conditions (SOCs). The Hessian matrix*

$$H(x_1, x_2) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite, together with the linearity of the restriction guarantees that the solution minimizes f.

The geometry of the problem is depicted in figure 1. The gradient of f and the gradient of the restriction at the optimum must have the same direction, although different lengths. In particular,

$$\nabla f(x^*) = \lambda^* \nabla g(x^*)$$

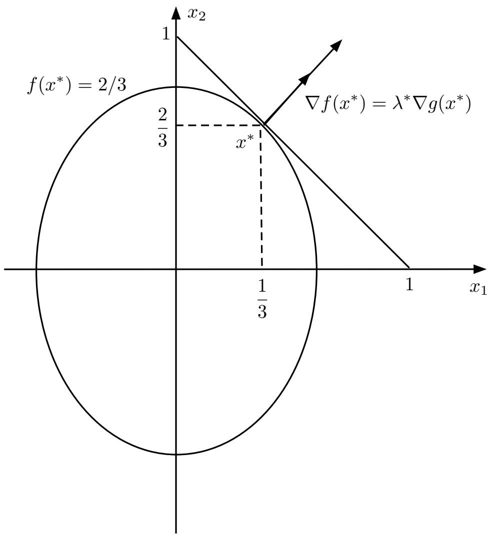

Figure 1: Problem 4.1

4.2 Suppose we have a distribution center that distributes goods to several retail outlets in a city. There are two routes to go from the distribution center to the city A and B. The cost of shipping x units using route A is $ax^2, a > 0$. The cost of shipping y units using route B is $by^2, b > 0$.

(a) Suppose Q units have to be distributed. Determine how they must be allocated to routes A and B to minimize the total shipping cost.
Solution: *The problem to solve is*

$$\min_{x,y} ax^2 + by^2 \text{ s.t.}$$
$$x + y = Q$$

The Lagrangean function is

$$L(x, y, \lambda) = ax^2 + bY^2 + \lambda(Q - x - y)$$

The first-order conditions (FOCs) are

$$\frac{\partial L}{\partial x} = 2ax - \lambda = 0$$
$$\frac{\partial L}{\partial y} = 2by - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = Q - x - y = 0$$

From the first two equations we obtain, $x = \frac{b}{a}y$. *Substituting in the third equation gives the solution:*

$$x^* = \frac{bQ}{a+b}, \quad y^* = \frac{aQ}{a+b}, \quad \lambda^* = \frac{2abQ}{a+b}, \quad f(x^*, y^*) = \frac{abQ^2}{a+b}.$$

The Hessian matrix

$$H(x,y) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

is positive definite since $a > 0, b > 0$. *Accordingly, the solution* (x^*, y^*) *minimizes the cost.*

(b) How does the cost change if Q increases by r%?

Solution: *If* Q *increases by* $r\%$, *the constraint increases by* $\Delta = rQ$ *and the minimum cost increases by* $\lambda^*\Delta = \frac{2abrQ^2}{a+b}$. *In other words the minimum cost increases by* $2r\%$.

4.3 An individual has some savings that wants to invest. He wants to minimize risk and obtain an expected return of 12%. There are three mutual funds available yielding expected returns of 10%, 10%, and 15% respectively. Let x, y, and z be the proportion of the savings invested in each of the three funds. The financial experts report that the measure of risk is given by

$$400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$$

Determine how the individual should distribute his savings among the three funds minimizing the risk.

Solution: *The problem to solve is*

$$\min_{x,y,z} 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz \text{ s.t.}$$
$$x + y + 1.5z = 1.2$$
$$x + y + z = 1$$

The Lagrangean function is

$$L(x,y,z,\lambda) = 400x^2+800y^2+200xy+1600z^2+400yz+\lambda_1(1.2-x-y-1.5z)+\lambda_2(1-x-y-z)$$

The first-order conditions (FOCs) are

$$\frac{\partial L}{\partial x} = 800x + 200y - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial y} = 1600y + 200x + 400z - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial z} = 3200z + 400y - 1.5\lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = 1.2 - x - y - 1.5z = 0$$
$$\frac{\partial L}{\partial \lambda_2} = 1 - x - y - z = 0$$

Solving the system yields

$$x^* = 0.5, \quad y^* = 0.1, \quad z^* = 0.4, \quad \lambda_1^* = 1800, \quad \lambda_2^* = -1380$$

4.4 An individual has preferences defined over three consumption goods x, y, z. This preferences are represented by means of an utility function

$$U(x, y, z) = 5 \ln x + 8 \ln y + 12 \ln z$$

Unit prices of the goods are $p_1 = 10$€, $p_2 = 15$€, $p_3 = 30$€. The income of the individual is $m = 3000$€.

Find the consumption bundle maximizing the utility of the individual.

Solution: *The problem to solve is*

$$\min_{x,y,z} 5 \ln x + 8 \ln y + 12 \ln z \textit{ s.t.}$$
$$10x + 15y + 30z = 3000$$

The Lagrangean function is

$$L(x, y, z, \lambda) = 5 \ln x + 8 \ln y + 12 \ln z + \lambda(3000 - 10x - 15y - 30z)$$

The first-order conditions (FOCs) are

$$\frac{\partial L}{\partial x} = \frac{5}{x} - 10\lambda = 0$$
$$\frac{\partial L}{\partial y} = \frac{8}{y} - 15\lambda = 0$$
$$\frac{\partial L}{\partial z} = \frac{12}{z} - 30\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 3000 - 10x - 15y - 30z = 0$$

From the first two FOCs we obtain

$$y = \frac{16}{15}x$$

From the first and the third FOCs we obtain

$$z = \frac{4}{5}x$$

Substituting these values into the constraint we obtain

$$10x + (15)\frac{16}{15}x + (30)\frac{4}{5}x = 50x = 3000$$

Therefore the utility maximizing bundle is given by

$$x^* = 60, \quad y^* = 64, \quad z^* = 48$$

Finally, form the first three FOCS we obtain

$$\lambda = \frac{1}{2x} = \frac{8}{15y} = \frac{2}{5z}$$

so that $\lambda^* = 1/120$.

4.5 A firm uses three inputs, u, v, w, to produce a certain good. Its production function is

$$Q(u, v, w) = 36u^{1/2}v^{1/3}w^{1/4}$$

The unit prices of the inputs are $p_u = 25$€, $p_v = 20$€, $p_w = 10$€.

(a) Find the levels of the inputs maximizing the output, given that the firm faces a budget constraint of $m = 78000$€

Solution: *The problem to solve is*

$$\min_{u,v,w} 36u^{1/2}v^{1/3}w^{1/4} \text{ s.t.}$$
$$25u + 20v + 10w = 78000$$

The Lagrangean function is

$$L(u, v, w, \lambda) = 36u^{1/2}v^{1/3}w^{1/4} + \lambda(78000 - 25u - 20v - 10w)$$

The first-order conditions (FOCs) are

$$\frac{\partial L}{\partial u} = 18u^{-1/2}v^{1/3}w^{1/4} - 25\lambda = 0$$
$$\frac{\partial L}{\partial v} = 12u^{1/2}v^{-2/3}w^{1/4} - 20\lambda = 0$$
$$\frac{\partial L}{\partial w} = 9u^{1/2}v^{1/3}w^{-3/4} - 10\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 78000 - 25u - 20v - 10w = 0$$

From the first two FOCs we obtain

$$v = \frac{5}{6}u$$

From the first and the third FOCs we obtain

$$w = \frac{5}{4}u$$

Substituting these values into the constraint we obtain

$$25u + (20)\frac{5}{6}u + (10)\frac{5}{4}u = 650U = 78000$$

Therefore the utility maximizing bundle is given by

$$u^* = 1440, \quad v^* = 1200, \quad w^* = 1800$$

Also,

$$\lambda^* = \frac{18(v^*)^{1/3}(w^*)^{1/4}}{25(u^*)^{1/2}} \approx 1.3133$$

and

$$Q^* \approx 94557.42$$

(b) Use the envelope theorem to assess how much can the firm increase the production if its budget increases to 80000€.

Solution: *By the envelope theorem we know that*

$$\frac{dQ^*}{dm} = \lambda^*$$

so by the approximation formula

$$\Delta Q^* \approx \lambda^* \Delta m = (1.3133)(2000) = 2662.6$$

Remark:
If we re-do the exercise assuming $m = 80000$ *we will obtain*

$$(u^*, v^*, w^*, \lambda^*) \approx (1476.92, 1230.77, 1846.15, 1.3161)$$

yielding $Q^* = 97186.8$ *so that* $\delta Q^* = 97186.80 - 94557.42 = 2629.38$
The error given by the approximation is of about 33 units or 1.2% *which can be considered as acceptable given the size of* Δm.

4.6 Let $f(x_1, x_2) = x_1 x_2$. Solve the following problem:

$$\min_{x_1, x_2} x_1 + x_2 \text{ s.t.}$$
$$x_1 + 4x_2 = 16$$

Solution: *The Lagrangian function is*

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(16 - x_1 - 4x_2)$$

The system of FOCs is

$$\frac{\partial L}{\partial x_1} = x_2 - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = x_1 - 4\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 16 - x_1 - 4x_2 = 0$$

From the first two equation we obtain $x_1 = 4x_2$. substituting it in the third FOC yields

$$16 - 4x_2 - 4x_2 = 0, \ or x_2 = 2 \Rightarrow (x_1 = 8, \ \lambda = 2)$$

To assess that the solution is actually minimizing the objective function f, we look at the second order conditions (SOCs). The Hessian matrix

$$H(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is positive definite, together with the linearity of the restriction guarantees that the solution minimizes f.

4.7 Let $f(x_1, x_2, x_3) = x_1 x_2 x_3, h_1(x, y, z) \equiv x_1^2 + x_2^2 = 1, h_2(x, y, z) \equiv x_1 + x_3 = 1$. Characterize the set of candidate solutions of the following problem:

$$\begin{aligned} \min_{x_1, x_2, x_3} \ & x_1 x_2 x_3 \text{ s.t.} \\ & x_1^2 + x_2^2 = 1 \\ & x_1 + x_3 = 1 \end{aligned}$$

Solution: *Let us start by verifying the constraint qualification. The Jacobian matrix of the constraints is*

$$Jh(x, y, z) = \begin{pmatrix} 2x_1 & 2x_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

This is singular only if $x_1 = x_2 = 0$. However, in such a case the restriction h_1 would be violated. Thus, we need not worry about this case and can look at the Lagrangean function:

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 x_2 x_3 - \lambda_1(x_1^2 + x_2^2 - 1) - \lambda_2(x_1 + x_3 - 1)$$

The system of FOCs is

$$\frac{\partial L}{\partial x_1} = x_2x_3 - 2\lambda_1 x_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_2} = x_1x_3 - 2\lambda_1 x_2 = 0$$
$$\frac{\partial L}{\partial x_3} = x_1x_2 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = x_1^2 + x_2^2 - 1 = 0$$
$$\frac{\partial L}{\partial \lambda_2} = x_1 + x_3 - 1 = 0$$

The third equation can be written as $\lambda_2 = x_1x_2$ *and the fifth equation can be rewritten as* $x_3 = 1 - x_1$ *Substituing them, the system of FOCs reduces to*

$$\frac{\partial L}{\partial x_1} = x_2(1 - x_1) - 2\lambda_1 x_1 - x_1x_2 = 0$$
$$\frac{\partial L}{\partial x_2} = x_1(1 - x_1) - 2\lambda_1 x_2 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = x_1^2 + x_2^2 - 1 = 0$$

From the second equation we obtain $2\lambda_1 = \frac{x_1(1-x_1)}{x_2}$ *that is well-defined as long as* $x_2 \neq 0$*.*

Case 1: $x_2 \neq 0$ *Substituting the value of* λ_2 *into the first equation, we obtain*

$$x_2^2(1 - 2x_1) = x_1^2(1 - x_1)$$
$$x_1^2 + x_2^2 = 1$$

From the second equation $x_2^2 = 1 - x_1^2$ *and substituting it into the first one we obtain*

$$3x_1^3 - 2x_1^2 - 2x_1 + 1 = 0 \text{ or}$$
$$(1 - x_1)(-3x_1^2 - x_1 + 1) = 0$$

Note that this equation is satisfied if $x_1 = 0$*. But in turn it implies* $x_3 = 0$ *and* $x_2 = 0$ *thus violating the initial condition defining Case 1, namely* $x_2 \neq 0$*. Accordingly, this is not a candidate solution.*

The expression $(-3x_1^2 - x_1 + 1)$ *equals zero when* $x_1 = \dfrac{-1 \pm \sqrt{13}}{6} \approx \{0.4343, -0.7676\}$*. Then,*

$$x_1 \approx 0.4343 \Rightarrow x_2 \approx \pm 0.9008, x_3 \approx 0.5657$$
$$x_1 \approx -0.7676 \Rightarrow x_2 \approx \pm 0.6409, x_3 \approx 1.7676$$

so we have obtained four candidate solutions in Case 1.

Case 2: $x_2 = 0$ *When* $x_2 = 0$ *we obtain*

(a) $x_1 = 1, x_3 = 0$

(b) $x_1 = -1, x_3 = 2$

The values $x_1 = -1, x_3 = 2$ *violate the FOC corresponding to* $\frac{\partial L}{\partial x_2}$ *and thus cannot be a candidate equilibrium. Thus Case 2 contributes with an additional solution candidate.*

We conclude that the problem has five candidate solutions. The examination of SOCs would elicit which are solutions of the problem.