Optimization. A first course on mathematics for economists
Problem set 2: Continuity

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2.1 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x \sin x$. Show that f is continuous.

Solution: *We know x and $\sin x$ are continuous and f is the product of continuous functions, thus continuous.*

2.2 Let $f : \mathbb{R} \to \mathbb{R}^2$ be continuous. Show that $g(x) = f(x^2 + x^3)$ is continuous.

Solution: *Function g is the composition of f on the continuous function $x \to x^2 + x^3$, thus continuous.*

2.3 Let $f(x) = \dfrac{x^2}{1+x}$. Find the points where f is continuous.

Solution: *Define f for $x \neq -1$. Then, f is the quotient of two continuous functions, thus continuous*

2.4 Find the sets of points where the following functions are continuous.

(i) $f(x) = x \sin(x^2)$
Solution: *Everywhere*

(ii) $f(x) = \dfrac{x + x^2}{x^2 - 1},\ x^2 \neq 1,\ f(\pm 1) = 0$
Solution: *f is continuous on $\mathbb{R} \setminus \{1, -1\}$*

(iii) $f(x) = \dfrac{\sin x}{x},\ x \neq 0,\ f(0) = 1$
Solution: *Everywhere*

Solution:

2.5 Let $A = \{x \in \mathbb{R} | \sin x = 0.56\}$. Show that A is a closed set. Is it compact?

Solution: *Note that $\{0.56\}$ is closed and $\sin x$ is continuous. Then, A is closed but it is not compact.*

2.6 Show $f : \mathbb{R} \to \mathbb{R}, x \to \sqrt{|x|}$ is continuous

Solution: *Define $g(x) = |x|$ and $h(x) = \sqrt{x}$. Both g and h are continuous functions. Then, $f = g \circ h$ and thus continuous.*

2.7 Show $f(x) = \sqrt{x^2 + 1}$ is continuous

Solution: *Define $g(x) = \sqrt{x}$ and $h(x) = x^2 + 1$. Both g and h are continuous functions. Then, $f = g \circ h$ and thus continuous.*

2.8 Let $f(x)$ be a cubic polynomial. Argue that f has a real root.

Solution: *Consider $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$. We know that f is continuous. Suppose $a > 0$. For $x > 0$ sufficiently large, $ax^3 > 0$ and will induce $f(x) > 0$. Similarly, for $x < 0$ sufficiently large, $ax^3 < 0$ and will induce $f(x) < 0$. Hence, applying the intermediate value therorem we conclude that $\exists x_0$ such that $f(x_0) = 0$.*
Remark: This statement can be applied to any odd-degree polynomial. However it does not apply to even-degree polynomials.

2.9 Let $f : [1, 2] \to [0, 3]$ be a continuous function with $f(1) = 0, f(2) = 3$. Show that f has a fixed point in $[1, 2]$.

Solution: *Define $g(x) = f(x) - x$. Then, g is continuous because is the difference of two continuous functions. Also, $g(1) = -1$ and $g(2) = 1$. Hence, applying the intermediate value theorem we conclude that $\exists x_0$ such that $g(x_0) = 0$. Therefore, $g(x_0) = 0 = f(x_0) - x_0$. That is $f(x_0) = x_0$, and x_0 is a fixed point of f.*