# Optimization. A first course on mathematics for economists
# Problem set 3: Differentiability

Xavier Martinez-Giralt

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3.1 Let $f(x, y) = x^2 y$

(a) Find $\nabla f(3, 2)$
**Solution**: *The gradient is the vector of partial derivatives. The partial derivatives of $f$ at the point $(x, y) = (3, 2)$ are:*

$$\frac{\partial f}{\partial x}(x, y) = 2xy \implies \frac{\partial f}{\partial x}(3, 2) = 12$$
$$\frac{\partial f}{\partial y}(x, y) = x^2 \implies \frac{\partial f}{\partial x}(3, 2) = 9$$

*Therefore, the gradient is*

$$\nabla f(3, 2) = (12, 9)$$

(b) Find the derivative of $f$ in the direction of $u = (1, 2)$ at the point $(3, 2)$.
**Solution**: *To compute a directional derivative first we need to compute the unit vector $e = (e_1, e_2)$. Given the direction $u = (1, 2)$, the length of this vector is*

$$\|u\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

*Then,*

$$e = (e_1, e_2) = \frac{u}{\|u\|} = \Big(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big)$$

*The directional derivative requested is*

$$\nabla f(3, 2) \cdot (e_1, e_2)^T = (12, 9) \cdot \Big(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big)^T = \frac{12}{\sqrt{5}} + \frac{18}{\sqrt{5}} = \frac{30}{\sqrt{5}}$$

(c) Find the derivative of $f$ in the direction of $u = (2, 1)$ at the point $(3, 2)$.
**Solution**: *To compute a directional derivative first we need to compute the unit vector $e = (e_1, e_2)$. Given the direction $u = (2, 1)$, the length of this vector is*

$$\|u\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

*Then,*

$$e = (e_1, e_2) = \frac{u}{\|u\|} = \Big(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\Big)$$

*The directional derivative requested is*

$$\nabla f(3,2) \cdot (e_1, e_2)^T = (12, 9) \cdot \Big(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\Big)^T = \frac{24}{\sqrt{5}} + \frac{9}{\sqrt{5}} = \frac{33}{\sqrt{5}}$$

(d) Identify in which direction is the directional derivative maximal at the point $(3,2)$. What is the directional derivative in that direction?

**Solution**: *The gradient points in the direction of the maximal directional derivative. Therefore, at the point* $(3,2)$ *the directional derivative is maximal in the direction of* $(12,9)$.
*In this direction, the unit vector is*

$$e = (e_1, e_2) = \frac{u}{\|u\|} = \Big(\frac{12}{15}, \frac{9}{15}\Big) = \Big(\frac{4}{5}, \frac{3}{5}\Big)$$

3.2 Let $f(x,y,z) = xye^{x^2+z^2-5}$. Calculate the gradient of $f$ at the point $(1,3,-2)$ and calculate the directional derivative at the point $(1,3,-2)$ in the direction of the vector $u = (3,-1,4)$.

**Solution**: *To compute the gradient we need to compute the partial derivatives of $f$:*

$$\frac{\partial f}{\partial x}(x,y,z) = (y + 2x^2y)e^{x^2+z^2-5} \Longrightarrow \frac{\partial f}{\partial x}(1,3,-2) = 3 + 2(3)(1) = 9$$
$$\frac{\partial f}{\partial y}(x,y,z) = xe^{x^2+z^2-5} \Longrightarrow \frac{\partial f}{\partial y}(1,3,-2) = 1(1) = 1$$
$$\frac{\partial f}{\partial z}(x,y,z) = 2xyze^{x^2+z^2-5} \Longrightarrow \frac{\partial f}{\partial z}(1,3,-2) = 2(1)(3)(-2)(1) = -12$$

*so that* $\nabla f(1,3,-2) = (9,1,-12)$.
*Next we have to compute the unit vector* $e = (e_1, e_2, e_3)$. *Given the direction* $u = (3,-1,4)$, *the length of this vector is*

$$\|u\| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$$

*so that*

$$e = (e_1, e_2, e_3) = \frac{u}{\|u\|} = \Big(\frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}}\Big)$$

*Finally, the directional derivative requested is*

$$\nabla f(1,3,-2) \cdot (e_1, e_2, e_3)^T = (9,1,-12) \cdot \Big(\frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}}\Big)^T =$$
$$\frac{27}{\sqrt{26}} + \frac{-1}{\sqrt{26}} + \frac{-48}{\sqrt{26}} = \frac{-22}{\sqrt{26}}.$$

3.3 Consider an industry producing a consumption good supplied according to the following supply function $S = S(w, p)$ where $w$ represents the wage rate and $p$ the price. Also, demand for the consumption good is captured by the demand function $D = D(m, p)$ where $m$ denotes income. Assume

$$\frac{\partial S}{\partial p} > 0, \quad \frac{\partial S}{\partial w} < 0$$
$$\frac{\partial D}{\partial p} < 0, \quad \frac{\partial D}{\partial m} > 0$$

Assess how a change in the wage rate $w$ and in the income $m$ affects the equilibrium price.

**Solution**: *The equilibrium condition is given by*

$$z(w, m, p) = S(w, p) - D(m, p) = 0 \tag{1}$$

*The question to be answered is the sign of* $\frac{\partial p}{\partial m}$ *and* $\frac{\partial p}{\partial w}$.
*Note that*

$$\frac{\partial z}{\partial p} = \frac{\partial S}{\partial p} - \frac{\partial D}{\partial p} > 0$$

*so this equation determines the price* $p$ *as a function of income* $m$ *and wage rate* $w$ *around the equilibrium point.*
*Compute the partial derivatives of* (1) *with respect to* $w$ *and* $m$*:*

$$\begin{aligned} \frac{\partial S}{\partial p}\frac{\partial p}{\partial m} - \frac{\partial D}{\partial p}\frac{\partial p}{\partial m} - \frac{\partial D}{\partial m} =& 0 \\ \frac{\partial S}{\partial p}\frac{\partial p}{\partial w} + \frac{\partial S}{\partial w} - \frac{\partial D}{\partial p}\frac{\partial p}{\partial w} =& 0 \end{aligned}$$

*Rearranging, we obtain*

$$\begin{aligned} \left(\frac{\partial S}{\partial p} - \frac{\partial D}{\partial p}\right)\frac{\partial p}{\partial m} =& \frac{\partial D}{\partial m} \\ \left(\frac{\partial S}{\partial p} - \frac{\partial D}{\partial p}\right)\frac{\partial p}{\partial w} =& -\frac{\partial S}{\partial w} \end{aligned}$$

*so that*

$$\begin{aligned} \frac{\partial p}{\partial m} =& \frac{\frac{\partial D}{\partial m}}{\frac{\partial S}{\partial p} - \frac{\partial D}{\partial p}} > 0 \\ \frac{\partial p}{\partial w} =& \frac{-\frac{\partial S}{\partial w}}{\frac{\partial S}{\partial p} - \frac{\partial D}{\partial p}} > 0 \end{aligned}$$

*Therefore, the price increases with both an increase in income and wage.*

3.4 Verifiy the homogeneity of

$$f(x_1, x_2, x_3, x_4) = \frac{x_1 + 2x_2 + 3x_3 + 4x_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

**Solution**: *Multiply all variables by t to obtain*

$$f(tx_1, tx_2, tx_3, tx_4) = \frac{tx_1 + 2tx_2 + 3tx_3 + 4tx_4}{(tx_1)^2 + (tx_2)^2 + (tx_3)^2 + (tx_4)^2} = \frac{t(x_1 + 2x_2 + 3x_3 + 4x_4)}{t^2(x_1^2 + x_2^2 + x_3^2 + x_4^2)} = t^{-1} f(x_1, x_2, x_3, x_4)$$

*so that $f$ is homogeneous of degree -1.*

3.5 Consider a general Cobb-Douglas production function

$$f(x_1, \ldots, x_n) = A \prod_{i=1}^{n} x_i^{a_i}$$

(a) Show that it is homogeneous.

**Solution**: *Let $t > 0$ and define $b = \sum_{i=1}^{n} a_i$. Now compute*

$$f(tx) = A \prod_{i=1}^{n} (tx_i)^{a_i} = At^b \prod_{i=1}^{n} x_i^{a_i} = At^b f(x)$$

*so that $f$ is homogeneous of degree $b$.*

(b) Determine when it has constant, decreasing, or increasing returns to scale.

**Solution**: *Constant returns to scale: $b = 1$; Increasing returns to scale: $b > 1$; Decreasing returns to scale: $b < 1$.*

3.6 Show that the constant elasticity of substitution (CES) function

$$f(x) = A\Big(\sum_{i=1}^{n} \delta_i x_i^{-\rho}\Big)^{-v/\rho}$$

where $A > 0, v > 0, \delta_1 > 0, \sum_i \delta_i = 1, \rho > -1, \rho \neq 0$, is homogeneous of degree $v$

**Solution**: *Let $t > 0$ and compute*

$$f(tx) = A\Big(\sum_{i=1}^{n} \delta_i (tx_i)^{-\rho}\Big)^{-v/\rho} = A\Big(t^{-\rho} \sum_{i=1}^{n} \delta_i x_i^{-\rho}\Big)^{-v/\rho} = t^v A\Big(\sum_{i=1}^{n} \delta_i x_i^{-\rho}\Big)^{-v/\rho} = t^v f(x)$$

3.7 Consider an individual consuming two goods $(x, y)$ available at prices $(p_x, p_y)$. The individual determines the demand of each good given those prices and the income $m$ defining the budget constraint $m = p_x x + p_y y$. Denote the resulting demands by $x(p_x, p_y, m)$ and $y(p_x, p_y, m)$ Show that these demands are homogeneous of degree zero in prices and income.

**Solution**: *Consider the demand of good $x$. Suppose all prices and income are multiplied by a factor $t$. The first observation is that the budget constraint is unaffected by such factor:*

$$m = p_x x + p_y y \Longleftrightarrow tm = tp_x x + tp_y y$$

*Accordingly, the demand of good $x$ is not affected by the factor $t$:*

$$x(p_x, p_y, m) = x(tp_x, tp_y, tm)$$

*Thus the demand of good $x$ is homogeneous of degree zero in prices and income. The same argument applies to de demand of good $y$.*

3.8 Approximate $\sqrt{5}$ to at least accuracy $1/100$ around $x = 4$.

**Solution**: *Consider $f(x) = \sqrt{x}$ and compute*

$$\frac{\partial f}{\partial x} = \frac{1}{2}x^{-1/2}$$
$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4}x^{-3/2}$$

*The second-order Taylor approximation of $f$ around $x = 4$ is*

$$P_2(x) = f(4) + f'(4)(x-4) + \frac{1}{2!}f''(4)(x-4)^2 =$$
$$2 + \frac{1}{2}4^{-1/2}(x-4) + \frac{1}{2!}(-\frac{1}{4})4^{-3/2}(x-4)^2 =$$
$$2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

*Evaluate $P_2(x)$ at $x = 5$ to obtain $P_2(5) = \frac{143}{64} \approx 2.234375$*
*Taylor's theorem tells us that the measurement error is given by*

$$|f(x) - P_2(x)| \leq \frac{1}{3!}M|x-4|^3,$$

*where $M \leq |f'''(x)|$.*
*Computing the third derivative we obtain $f'''(x) = \frac{3}{8}x^{-5/2}$. This is a decreasing function of $x$. Thus, in the interval $[4, 5]$ the maximum of $f'''$ is achieved at $x = 4$. Then, $f'''(4) = \frac{3}{8}\frac{1}{32} = \frac{3}{256}$ and*

$$|f(x) - P_2(x)| \leq \frac{1}{3!}\frac{3}{256} = \frac{1}{512}.$$

*Therefore, the approximation given by $P_2(5) \approx 2.234375$ is guaranteed to be accurate to within at least $\frac{1}{512}$ that is less than $1/100$.*