# Optimization. A first course on mathematics for economists
# Problem set 6: Linear programming

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6.1 A company produces two goods $x$ and $y$. The production technology for each good is described as follows

- one unit of good $x$ requires
  - $4m^2$ of storage space
  - 5 units of raw materials
  - 1 minute of production time
- one unit of good $y$ requires
  - $5m^2$ of storage space
  - 3 units of raw materials
  - 2 minutes of production time

The company has

- premises of $1500m^2$ to store the products before their distribution,
- 1575 units of raw materials daily,
- works 7 hours per day,
- at the end of the day the whole production is shipped out.

Finally, the selling unit prices of the goods are 13€ and 11€ for good $x$ and $y$ respectively.

(a) Formulate the revenue maximizing daily production of goods $x$ and $y$

(b) Solve the problem graphically,

(c) Solve the problem analytivally,

(d) Solve the problem usim the simplex algorithm,

(e) Formulate the dual problem

**Solution**:

(a) Let $x$ and $y$ denote (abusing notation) the daily output of both products. The associated revenue function is

$$R(x,y) = 13x + 11y$$

The restrictions induced by the technology are

- storage capacity: $4x + 5y \leq 1500$
- raw materials: $5x + 3y \leq 1575$
- production time (in minutes): $x + 2y \leq 420$
- also, for consistency, $x \geq 0$ and $y \geq 0$

Therefore, the problem the company faces is

$$\max_{x,y} R(x,y) = 13x + 11y \text{ s.t.} \tag{1}$$
$$4x + 5y \leq 1500 \tag{2}$$
$$5x + 3y \leq 1575 \tag{3}$$
$$x + 2y \leq 420 \tag{4}$$
$$x \geq 0, \;\; y \geq 0 \tag{5}$$

(b) Figure 1 shows the graphical representation of the problem: The pro-

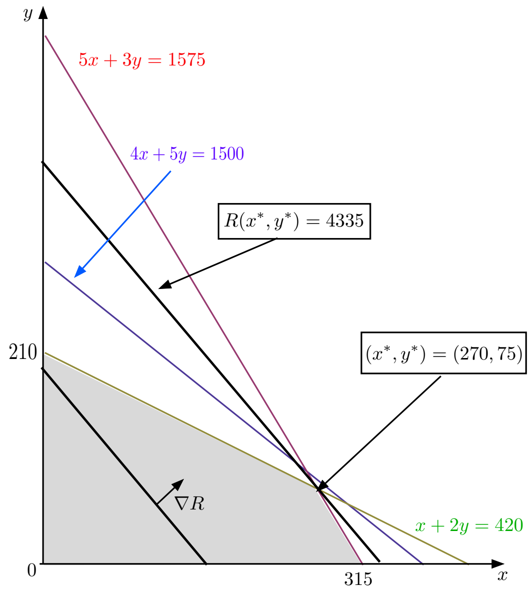


Figure 1: Problem 6.1

duction bundle maximizing revenue is $(x^*, y^*) = (270, 75)$ daily units and the associated level of revenues $R(x^*, y^*) = 4335$€

Note that at the solution, restriction (2) (in blue) is not binding.

(c) To solve the problem analytically, recall that the solution of linear programing problem when it exists, may be either at a vertex of the feasible set, or along a section of its frontier.

Let us star computing the slopes of the level sets of the revenur functionand of the restrictions:

- Consider a representative level set of the revenue function $13x + 11y = k$. Totally differentiating we obtain its slope given by $dy/dx = -13/11$
- Consider the frontier of restriction (2). Its slope is $dy/dx = -4/5$.
- Consider the frontier of restriction (3). Its slope is $dy/dx = -5/3$.
- Consider the frontier of restriction (4). Its slope is $dy/dx = -1/2$.

Accordingly, if a solution exists it will be locates at a vertex of the feasible set. Those vertices are defined by the intersection of (at least) two of the restriction of the problem. Let,

$$
\begin{aligned}
g_1(x, y) &\equiv 1500 - 4x - 5y = 0 \\
g_2(x, y) &\equiv 1575 - 5x - 3y = 0 \\
g_3(x, y) &\equiv 420 - x - 2y = 0
\end{aligned}
$$

Then,

- $g_1 \cap g_3 \rightarrow (x, y) = (60, 300)$ But $(x, y) = (60, 300) \notin g_2$ so that this vertex does not belong to the feasible set.
- $g_1 \cap g_2 \rightarrow (x, y) = (\frac{1200}{13}, \frac{3375}{13})$ But $(x, y) = (\frac{1200}{13}, \frac{3375}{13}) \notin g_3$ so that this vertex does not belong to the feasible set.
- $g_2 \cap g_3 \rightarrow (x, y) = (270, 75)$. Also, $(x, y) = (270, 75)$ belongs to the interior of (3) so that this vertex does belong to the feasible set.
- $g_1 \cap g_2 \cap g_3 = \emptyset$
- $(x \geq 0) \cap (y \geq 0) = (0, 0)$

From these intersections we conclude that the feasible set has an interior vertex at $(x, y) = (270, 75)$, and also a vertex at the origin.

Next we look at the vertices in the axes. These are defined by $\min\{g_1(0, y), g_2(0, y), g_3(0, y)\}$ on the $y$-axes and similarly by $\min\{g_1(x, 0), g_2(x.0), g_3(x, 0)\}$ on the $x$-axis:

$$
\begin{aligned}
g_1(0, y) &\rightarrow (x, y) = (0, 300) \\
g_2(0, y) &\rightarrow (x, y) = (0, 525) \\
g_3(0, y) &\rightarrow (x, y) = (0, 210) \\
g_1(x, 0) &\rightarrow (x, y) = (375, 0) \\
g_2(x, 0) &\rightarrow (x, y) = (315, 0) \\
g_3(x, 0) &\rightarrow (x, y) = (420, 0)
\end{aligned}
$$

Accordingly,

$$\min\{g_1(0,y), g_2(0,y), g_3(0,y)\} = (0, 210)$$
$$\min\{g_1(x,0), g_2(x.0), g_3(x,0)\} = (315, 0).$$

Summarizing, the vertices of the feasible set are

$$\{(0,0), (0,210), (270,75), (315,0)\}.$$

Finally, the only computation left is to evaluate $R(x,y)$ at each of the vertices and choose the production plan yielding the highest revenue:

$$R(0,0) = 0$$
$$R(0,210) = 2310$$
$$R(270,75) = 4335$$
$$R(315,0) = 4095$$

We thus conclude (in consistency with the graphical analysis) that the production bundle maximizing revenue is $(x^*, y^*) = (270, 75)$ daily units.

(d) The simplex algorithm proceeds in a sequence of steps by jumping from one vertex to another as long as there is a direction in which the objective function increases its value. The problem is described by equations (1)-(5).

**Step 1** : Add slack variables in the constraints

$$4x + 5y + s_1 = 1500 \tag{6}$$
$$5x + 3y + s_2 = 1575 \tag{7}$$
$$x + 2y + s_3 = 420 \tag{8}$$
$$x \geq 0,\ y \geq 0, s_1 \geq 0, (i = 1, 2, 3) \tag{9}$$

Now we will be dealing with five variables $(x, y, s_1, s_2, s_3)$. At each step some will be zero (non-basic variables), some others will be positive (basic variables).

**Step 2** : Choose a feasible vertex to start. Let's choose, $(x, y) = (0, 0)$. Compute the associated values of $(s_1, s_2, s_3)$ by substituting in (6)-(8): $s_1 = 1500, s_2 = 1575, s_3 = 420$. Hence the vector of variables is $(x, y, s_1, s_2, s_3) = (0, 0, 1500, 1575, 420)$. Also compute the value of the objective function: $R(0, 0) = 0$.

**Step 3** : Re-write the restrictions and the objective value in terms of the non-basic variables:

$$s_1 = 1500 - 4x - 5y \tag{10}$$
$$s_2 = 1575 - 5x - 3y \tag{11}$$
$$s_3 = 420 - x - 2y \tag{12}$$
$$R = 13x + 11y \tag{13}$$

**Step 4** : Determine the maximum feasible increases of $x$ and $y$:

- Consider $x$
  - in equation (10), the maximum increase in the value of $x$ is $\Delta x = \frac{1500}{4} = 375$
  - in equation (11), the maximum increase in the value of $x$ is $\Delta x = \frac{1575}{5} = 315$
  - in equation (12), the maximum increase in the value of $x$ is $\Delta x = 420$

  Therefore, the maximum *feasible* increase in the direction $x$ is the $\min\{375, 315, 420\} = 315$. In this direction, the associated increase in $R$ is $\Delta R = (13)(315) = 4095$.
- Consider $y$
  - in equation (10), the maximum increase in the value of $y$ is $\Delta y = \frac{1500}{3} = 300$
  - in equation (11), the maximum increase in the value of $y$ is $\Delta y = \frac{1575}{3} = 525$
  - in equation (12), the maximum increase in the value of $y$ is $\Delta y = 210$

  Therefore, the maximum *feasible* increase in the direction $y$ is the $\min\{300, 525, 210\} = 210$. In this direction, the associated increase in $R$ is $\Delta R = (11)(210) = 2310$.

The maximum increase in $R$ is obtained in the direction of $x$. Accordingly, we look for a next vertex moving along the $x$-axis. This next vertex is given by $(x + \Delta x, y) = (315, 0)$. At this new vertex, we have to compute the associated values of $(s_1, s_2, s_3)$ by substituting $(x + \Delta x, y) = (315, 0)$ in (10)-(12) to obtain $s_1 = 240, s_2 = 0, s_3 = 105$ and $R = 4095$.
Note incidentally, that moving from the initial vertex $(x, y) = (0, 0)$ to the vertex $(x, y) = (315, 0)$, the value of $R$ has increased (from 0 to 4095).

**Step 5** : The algorithm goes back to step 3, by redefining the constraints and the objective functions in terms of the new non-basic variables, namely $(y, s_2)$.
Consider (11) and solve it for $(y, s_2)$, so it gives

$$x = 315 - \frac{3}{5}y - \frac{1}{5}s_2 \tag{14}$$

Substituting (14) in (10), (12) and (13) we obtain

$$s_1 = 240 - \frac{13}{5}y + \frac{4}{5}s_2 \tag{15}$$

$$s_3 = 105 - \frac{7}{5}y - \frac{1}{2}s_2 \tag{16}$$

$$R = 4095 + \frac{16}{5}y - \frac{1}{2}s_2 \tag{17}$$

**Step 6** : Repeat step 4 looking for the next vertex. Note that (17) can only increase its value moving along the direction of $y$. Therefore, we only need to compute the maximum feasible increase of $y$:

- in equation (14), the maximum increase in the value of $y$ is $\Delta y = (315)\frac{5}{3} = 525$
- in equation (15), the maximum increase in the value of $y$ is $\Delta y = (240)\frac{5}{13} \approx 92.3$
- in equation (16), the maximum increase in the value of $y$ is $\Delta y = (105)\frac{5}{7} = 75$

The maximum *feasible* increase in the direction $y$ is given by $\min\{525, 92.3, 75\} = 75$. In this direction, the associated increase in $R$ is $\Delta R = (75)\frac{16}{5} = 240$. Thus the new valur of the objective function will be $R + \Delta R = 4095 + 214 = 4335$.

The movement in the direction $y$ generates a new set of values of the variables $(x, s_1, s_1, s_3)$ obtained by substituting $y = 75$ into (14)-(16): $x = 270, s_1 = 45, s_3 = 0$. Thus, we have a new vertex $(x, y) = (270, 75)$ and slack varaibles $(s_1, s_2, s_3) = (45, 0, 0, )$. In turn, (and in consistency with the previous comoutation), the new value of $R$ is obtained substituting $y = 75$ in (17) yielding $R = 4335$.

**Step 7** : The algorithm goes back to step 3. Given the vector of variables $(x, y, s_1, s_2, s_3 = (270, 75, 45, 0, 0)$, we rewrite the constraints and the objective fucntion in terms of the new non-basic variables $(s_2, s_3)$. Consider (16) and solve it for $(s_2, s_3)$ to obtain

$$y = 75 - \frac{5}{7}s_3 - \frac{5}{14}s_2 \tag{18}$$

Substituting (18) into (14) and (16) we obtain

$$x = 270 + \frac{3}{7}s_3 + \frac{1}{70}s_2 \tag{19}$$

$$s_1 = 45 - \frac{13}{7}s_3 - \frac{11}{70}s_2 \tag{20}$$

$$\tag{21}$$

Also Substituting (18) into (17) yields

$$R = 4335 - \frac{16}{7}s_3 - \frac{131}{35}s_2 \tag{22}$$

Note that there is no direction of increase of $R$. Therefore, the algorithm stops and gives as solution that the objective function is maximized at the vertex $(x^*, y^*) = (270, 75)$ and the maximum value of the revenue function is $R(x^*, y^*) = 4335$.

(e) To derive the dual problem, consider the following situation.

Suppose the company decreases its production in one unit of each product $yx$ and $y$. Given the technology this decision frees up

- $4m^2$ of storage room rom good $x$, and $5m^2$ associated to good $y$;
- 5 units of raw materials from good $x$, and 3 units of raw materials from good $y$
- 1 minute of production time from good $x$, and 2 minutes of production time from good $y$

Selling these idle inputs at prices $z_1, z_2, z_3$ respectively, would generata revenues

- $4z_1 + 5z_2 + z_3$€ from good $x$
- $5z_1 + 3z_2 + 2z_3$€ from good $y$

Recall that one unit of $x$ generates 13€to the revenue of the company, and one unit of $y$ generaters 11€of revenue.

Therefore, the company would be willing to reduce the production of $x$ and $y$ in one unit if the sale of the inputs unused offsets the forgone revenues, that is if

$$4z_1 + 5z_2 + z_3 \geq 13$$
$$5z_1 + 3z_2 + 2z_3 \geq 11$$

Assume now that there is a buyer willing to purchase the whole inventory of the company. Naturally, this buyer will aim at minimizing the cost of the acquisition (subject to the constraints).

The value of all the resources of the company is

$$P(z_1, z_2, z_3) = 1500z_1 + 1575z_2 + 420z_3$$

Thus, the problem of that buyer is

$$\begin{aligned} \min_{y_1, y_2, y_3} \; & 1500z_1 + 1575z_2 + 420z_3 \text{ s.t.} \\ & 4z_1 + 5z_2 + z_3 \geq 13 \\ & 5z_1 + 3z_2 + 2z_3 \geq 11 \\ & z_1 \geq 0, z_2 \geq 0, z_3 \geq 0 \end{aligned}$$

Writing the primal problem in matrix form as

$$\max_{x,y} \begin{pmatrix} 13 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ s.t. } \begin{pmatrix} 4 & 5 \\ 5 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1500 \\ 1575 \\ 420 \end{pmatrix}$$

The dual problem is simply

$$\min_{z_1,z_2,z_3} \begin{pmatrix} 1500 & 1575 & 420 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \text{ s.t. } \begin{pmatrix} 4 & 5 & 1 \\ 5 & 3 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \geq \begin{pmatrix} 13 \\ 11 \end{pmatrix}$$

where

- for each restriction in the primal problem, we introduce a new variable $z_1, z_2, z_3$.
- $\max_{x,y}$ in the primal problem becomes a $\min_{z_1,z_2,z_3}$ in the dual problem.
- the vector of constants in the restrictions of the primal problem becomes the vector of coefficients of the objective variable in the dual problem.
- the vector of parameters of the objective function of the primal problem becomes the vector of constants of the restrictions in the dual problem.
- the matrix of coefficients of the restrictions in the primal problem becomes the transposed matrix of coefficients of the restrictions in teh dual problem.
- the variables of the objective function of the proimal problem become the variables of the restrictions of the dual problem.

6.2 Solve

$$\begin{aligned} &\min_{y_1,y_2} 6y_1 + 8y_2 \text{ s.t.} \\ &\quad 2y_1 + y_2 \geq 3 \\ &\quad y_1 + 2y_2 \geq 2 \\ &\quad y_1 \geq 0, \quad y_2 \geq 0 \end{aligned}$$

**Solution**:

- Start by verifying whether the solution lies in a vertex or along a restriction. To do it compare, we the slopes of the restrictions and of the level sets of the objective function:
  - the slope of the level sets is -3/4
  - the slope of the first restriction is -2

- the slope of the second restriction is -1/2

  Therefore, the solution is located at a vertex.

- Next let's identify the vertices of the feasible set.
  - vertices along the axes

$$
\begin{aligned}
g_1(0, y_2) &\rightarrow (0, 3) \\
g_1(y_1, 0) &\rightarrow (\frac{3}{2}, 0) \\
g_2(0, y_2) &\rightarrow (0, 1) \\
g_1(y_1, 0) &\rightarrow (2, 0) \\
(y_1 \geq 0) \cap (y_2 \geq 0 &\rightarrow (0, 0)
\end{aligned}
$$

  - Note that vertices $(\frac{3}{2}, 0), (0, 1), (0, 0)$ violate at least one restriction. Therefore, the feasible vertices on the axes are $(0, 3)$ and $(2, 0)$
  - interior vertex is the solution of

$$
\begin{aligned}
2y_1 + y_2 &= 3 \\
y_1 + 2y_2 &= 2
\end{aligned}
$$

    that gives $(y_1, y_2) = (\frac{4}{3}, \frac{1}{3})$
  - Evaluate the objective function at the feasible vertices

$$
\begin{aligned}
F(0, 3) &= 24 \\
F(2, 0) &= 12 \\
F(\frac{4}{3}, \frac{1}{3}) &= \frac{32}{3}
\end{aligned}
$$

    The solution is the vertex yielding the minimum value of $F$. This is $(y_1^*, y_2^*) = (\frac{4}{3}, \frac{1}{3})$

Figure 2 summarizes the discussion.

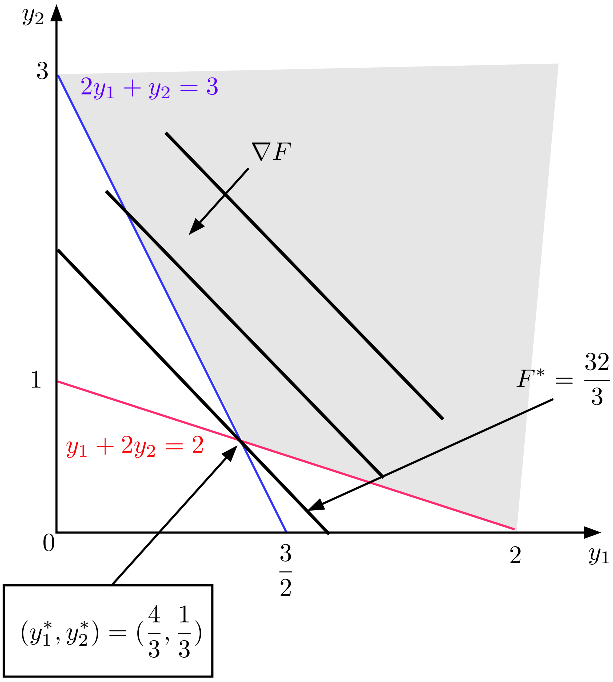


Figure 2: Problem 6.2