Optimization. A first course of mathematics for economists

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II.1 Static optimization - Introduction

Introduction

Economics

- Optimal allocation of scarce resources: efficiency, rational behavior.

The economic problem

- Instruments: variables
- Objective function: aim to be achieved
- Restrictions: scarce resources
- Opportunity set: set of instruments satisfying all restrictions

Economic problem: Choice of instruments in the feasible set allowing for optimizing objective function
Economic problem: particular case of general mathematical programming problem

Introduction (cont'd)

Formal definition of the problem

- Instruments: $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \cdots x_n \end{pmatrix}'$, $\mathbf{x} \in \mathbf{R^n}$
- Opportunity set: $X \subset \mathbf{R}^n$
- Objective function: $f : X \to \mathbf{R}^n$
- Problem: $max_{\mathbf{x}} f(\mathbf{x})$ s.t. $\mathbf{x} \in X$

Particular cases

- Classical programming
- Non-linear programming
- Linear programming

Classical programming - Definition

Equality restrictions

- $g_i(\mathbf{x}) = b_i, \; i = 1, \ldots, m$
- $g_i(\mathbf{x})$ continuous, continuously differentiable
- $b_i \in \mathbf{R}$
- $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix}$
- $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Problem:

- $\max_{\mathbf{x}} f(\mathbf{x})$ s.t. $\mathbf{g}(\mathbf{x}) = \mathbf{b}$

Nonlinear programming - Definition

Inequality restrictions

- $g_i(\mathbf{x}) \leq b_i,\ i = 1, \ldots, m$
- $g_i(\mathbf{x})$ continuous, continuously differentiable
- $b_i \in \mathbf{R}$

Non-negativity restrictions

- $x_j \geq 0,\ j = 1, \ldots, n$

Problem:

- $\max_{\mathbf{x}} f(\mathbf{x})$ s.t. $\begin{cases} \mathbf{g}(\mathbf{x}) \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{cases}$

Linear programming - Definition

Linear inequality restrictions

- $g_i(\mathbf{x}) = \sum_{i=1}^{m} a_{ij}x_j \leq b_i,\ i = 1, \ldots, m;\ j = 1, \ldots, n$
- $g_i(\mathbf{x})$ continuous, continuously differentiable, $b_i \in \mathbf{R}$
- $\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$

Non-negativity restrictions

- $x_j \geq 0,\ j = 1, \ldots, n$

Linear objective function

- $f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j = \mathbf{cx},\ j = 1, \ldots, n$
- $\mathbf{c} = \begin{pmatrix} c_1 \ldots c_n \end{pmatrix},\ c_j \in \mathbf{R}$

Problem:

- $\max_{\mathbf{x}} \mathbf{cx}$ s.t. $\mathbf{Ax} \leq \mathbf{b},\ \mathbf{x} \geq 0$

Programming - Examples

Example 1

- Find among all rectangles with perimeter $2p > 0$, the one with maximum area.
- Solution:
  Let x and y denote the base and heigth of the rectangle.
  Then, $x = y = p/2$ defines the rectangle with maximum area.

Example 2

- Find among all isosceles triangles with perimeter $p = 1$, the one with maximum area.
- Solution:
  Let x and y denote the (equal) sides and base of the triangle.
  Then, $x = 1/3$ and $y = 1/3$ defines the triangle with maximum area.

On stationary points

Global extreme points

- Let $\mathbf{x}^* \in X$. Let $f : X \to \mathbf{R}^n$.
- We say that $\mathbf{x}^*$ is a global maximum of f if $\forall \mathbf{x} \in X,\ f(\mathbf{x}^*) \geq f(\mathbf{x})$.
- We say that $\mathbf{x}^*$ is a strict global maximum of f if $\forall \mathbf{x} \in X,\ f(\mathbf{x}^*) > f(\mathbf{x}),\ \mathbf{x}^* \neq \mathbf{x}$.

Local extreme points

- Let $\mathbf{x}^* \in X$. Let $f : X \to \mathbf{R}^n$ Define an open ball $B(\mathbf{x}^*, r)$, with r arbitrarily small.
- We say that $\mathbf{x}^*$ is a local maximum of f if $\forall \mathbf{x} \in X \cap B(\mathbf{x}^*, r),\ f(\mathbf{x}^*) \geq f(\mathbf{x})$.
- We say that $\mathbf{x}^*$ is a strict local maximum of f if $\forall \mathbf{x} \in X \cap B(\mathbf{x}^*, r),\ f(\mathbf{x}^*) > f(\mathbf{x}),\ \mathbf{x}^* \neq \mathbf{x}$.

A theorem

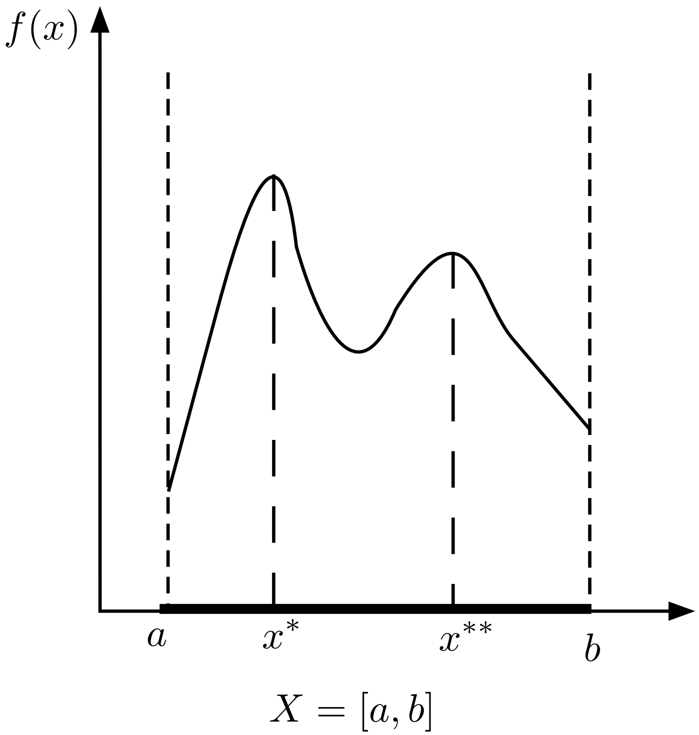
Weierstrass' extreme value theorem

- Let $X \subset \mathbf{R}$ be compact and non-empty.
- Let $f : X \to \mathbf{R}$ be continuous on X.
- Then, then f must attain a global maximum and a global minimum, each at least once. That is, $\exists (c, d) \in X$ such that $f(d) \leq f(x) \leq f(c), \; \forall x \in X$.

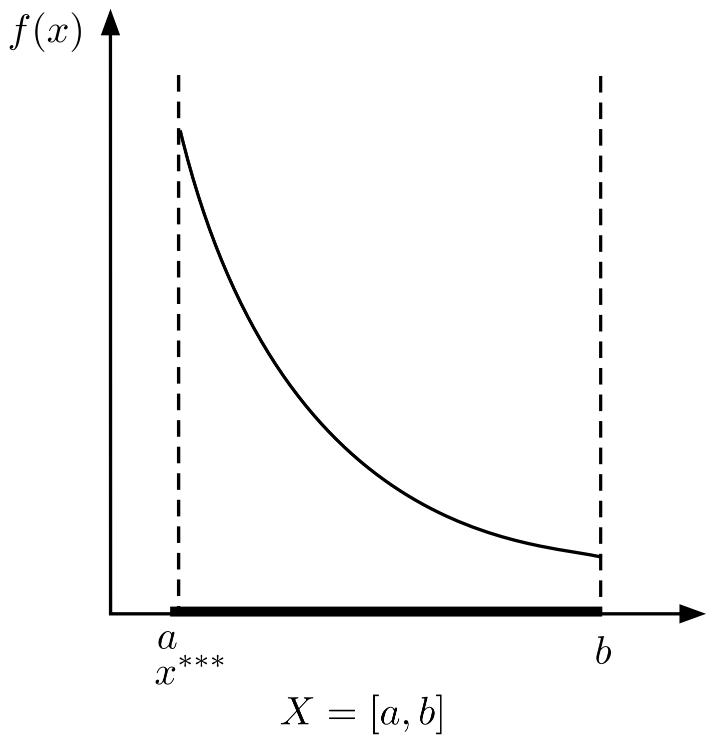
Remarks

- Theorem only allows to identify global extreme points.
- Theorem sufficient conditions, not necessary. [e.g. $f(x) = x^3, \; x \in (0, 1]$ has a maximum at $x = 1$ although $X = (0, 1]$ is not compact
- Local extreme points?

Weiertrass' theorem - Illustration



(a) Interior solutions



(b) Corner solution

On stationary points (2)

Introducing differentiability

- So far, only assumption on f is continuity.
- Some existence results, but no full characterization
- More structure on f is needed $\rightarrow$ differentiability.

On stationary points (3)

- Let $(\alpha, \beta) \subset X$. Let $f : X \to$ **R** be differentiable on (α, β).

Then

- If $f'(x) \geq 0 \; \forall x \in (\alpha, c)$ and $f'(x) \leq 0 \; \forall x \in (c, \beta)$, we say that $x = c$ is a local interior maximum point of f.
- If $f'(x) \leq 0 \; \forall x \in (\alpha, c)$ and $f'(x) \geq 0 \; \forall x (c, \beta)$, we say that $x = c$ is a local interior minimum point of f.
- If $f'(x) < 0 \; \forall x \in (\alpha, \beta)$, we say that $x = c$ is not an extreme point of f,
- If $f'(x) > 0 \; \forall x \in (\alpha, \beta)$, we say that $x = c$ is not an extreme point of f.

On stationary points (4)

Fermat's stationary points theorem

- Let $[a, b] \subset \mathbf{R}$, and let $x \in (a, b)$ be a local extremum of f
- Let $f : [a, b] \to \mathbf{R}$ be differentiable at x.
- Then, $f'(x) = 0$.

Remark

- Theorem only characterizes interior extreme points.
- Theorem does not allow to distinguish between maximum and minimum points.
- Also, a global extreme of f may also occur at
  - a non-differentiable point
  - a boundary point
- can we go beyond the "test of first derivative" ?

On stationary points (5)

Proposition

- Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b).
- Let $c \in (a, b)$ be a stationary point of f, i.e. $f'(c) = 0$.

Then

- If $f''(c) < 0$, we say that c is a local maximum point of f.
- If $f''(c) > 0$, we say that c is a local minimum point of f.

What if $f''(c) = 0$?

Inflection points

Remark

- $f''(x) = 0$ does not give information
- $f(x) = x^4 \to f'(0) = 0, f''(0) = 0$ and $x = 0$ is a minimum
- $f(x) = -x^4 \to f'(0) = 0, f''(0) = 0$ and $x = 0$ is a maximum
- $f(x) = x^3 \to f'(0) = 0, f''(0) = 0$ and $x = 0$ is a inflection point

Definition

- An inflection point: function concave $\leftrightarrow$ convex.
- Let $f : X \to$ **R** be twice differentiable. Let $(a, b, c) \in X,\ a < c < b$.
- We say that c is an inflection point of f if one of the following conditions holds:
  - If $f''(x) \geq 0\ \forall x \in (a, c)$ and $f''(x) \leq 0\ \forall x \in (c, b)$, or
  - If $f''(x) \leq 0\ \forall x \in (a, c)$ and $f''(x) \geq 0\ \forall x (c, b)$
- Remark: Definition does not require $f'(c) = 0$.

Inflection points (2)

Theorem

- Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b).
- Let $c \in (a, b)$.
- Then:
  - If c is an inflection point, $f''(c) = 0$
  - If $f'(c) = 0$ and f'' changes sign at c, c is a stationary inflection point
  - If $f'(c) \neq 0$ and f'' changes sign at c, c is a non-stationary inflection point

Concave and convex functions

Definitions

- Let $f : [a, b] \to \mathbf{R}$ be twice differentiable on (a, b).
- We say that f is convex on (a, b) iff $f''(x) \geq 0,\ \forall x \in (a, b)$
- We say that f is concave on (a, b) iff $f''(x) \leq 0,\ \forall x \in (a, b)$
- We say that f is strictly convex on (a, b) iff $f''(x) > 0,\ \forall x \in (a, b)$
- We say that f is strictly concave on (a, b) iff $f''(x) < 0,\ \forall x \in (a, b)$

Concavity, convexity and extreme points

Theorem

- Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b).
- Let $c \in (a, b)$ be such that $f'(c) = 0$.
- Then:
  - If $f''(c) \leq 0, \ \forall x \in (a, b)$ then $f(c) \geq f(x) \ \forall x \in [a, b]$
  - i.e. if f concave at c, then c is a (local) maximum point.
  - If $f''(c) \geq 0, \ \forall x \in (a, b)$ then $f(c) \leq f(x) \ \forall x \in [a, b]$
  - i.e. if f convex at c, then c is a (local) minimum point.

Concave and convex functions (2)

Preliminaries

- Let $f : [a, b] \to \mathbb{R}$ Remark: f need not be differentiable
- A point $\tilde{x} \in (a, b)$ can be written as $(1 - \lambda)a + \lambda b,\ \lambda \in [0, 1]$
- The equation of the segment joining a and b is given by

$$R(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

- Evaluating $R(x)$ at $\tilde{x}$,

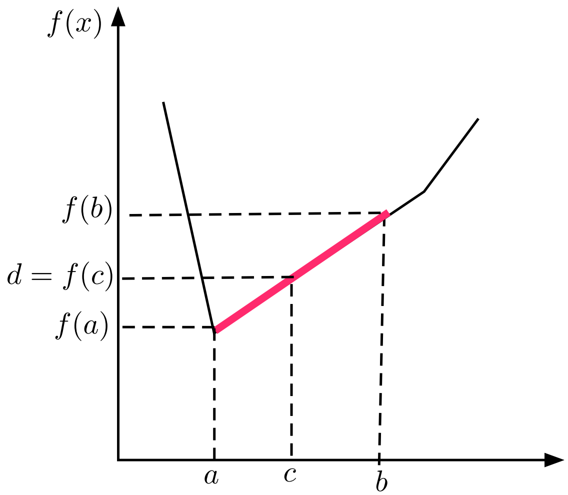
$$\begin{aligned} R(\tilde{x}) &= \frac{f(b) - f(a)}{b - a}((1 - \lambda)a + \lambda b - a) + f(a) \\ &= \frac{f(b) - f(a)}{b - a}(\lambda b - \lambda a) + f(a) \\ &= (1 - \lambda)f(a) + \lambda f(b) \end{aligned}$$

Concave and convex functions (3)

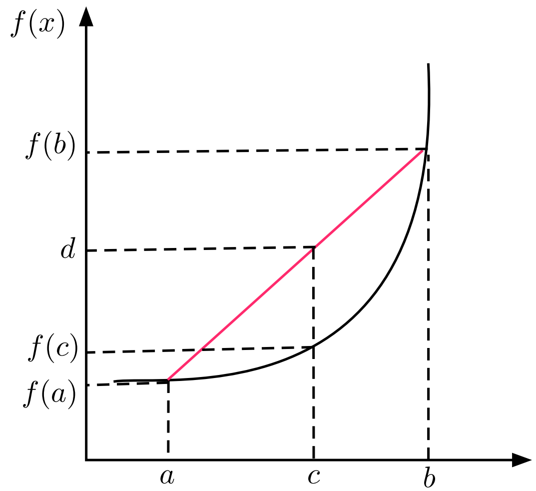
Definitions

- Let $f : A \subset \mathbf{R}^n \to \mathbf{R},\ A$ convex (No differentiability required)
- We say that f is concave on A iff $\forall (a,b) \in A$
  $(1-\lambda)f(a) + \lambda f(b) \leq f((1-\lambda)a + \lambda b)$ or $R(\tilde{x}) \leq f(\tilde{x})$
- We say that f is convex on A iff $\forall (a,b) \in A$
  $(1-\lambda)f(a) + \lambda f(b) \geq f((1-\lambda)a + \lambda b)$ or $R(\tilde{x}) \geq f(\tilde{x})$
- We say that f is strictly concave on A iff $\forall (a,b) \in A$
  $(1-\lambda)f(a) + \lambda f(b) < f((1-\lambda)a + \lambda b)$ or $R(\tilde{x}) < f(\tilde{x})$
- We say that f is strictly convex on A iff $\forall (a,b) \in A$
  $(1-\lambda)f(a) + \lambda f(b) > f((1-\lambda)a + \lambda b)$ or $R(\tilde{x}) > f(\tilde{x})$
- We say that f is quasi-concave on A iff $\forall (a,b) \in A$
  $\min\{f(a), f(b)\} \leq f((1-\lambda)a + \lambda b)$
- We say that f is strictly quasi-concave on A iff $\forall (a,b) \in A$
  $\min\{f(a), f(b)\} < f((1-\lambda)a + \lambda b)$

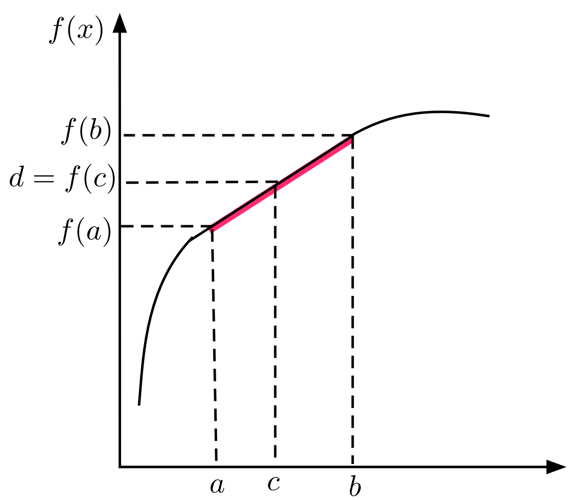
Convex and concave functions - Illustration



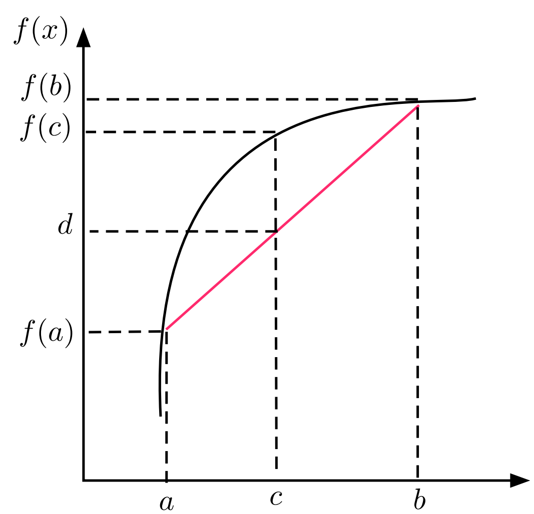
f convex



f strictly convex



f concave



f strictly concave

$$c = (1-\lambda)a + \lambda b$$

$$d = (1-\lambda)f(a) + \lambda f(b)$$

Concave and convex functions (4)

Definitions

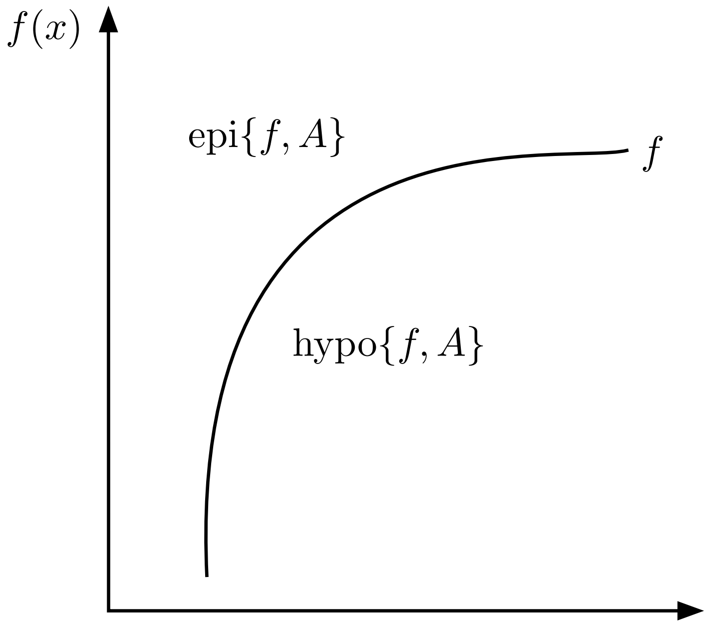
- Let $f : A \subset \mathbf{R}^n \rightarrow \mathbf{R}$
- The hypograph of f is the set
  hypo$\{f, A\} = \{(x, y) \in \mathbf{R}^{n+1} | x \in A, y \leq f(x)\}$
- The epigraph of f is the set
  epi$\{f, A\} = \{(x, y) \in \mathbf{R}^{n+1} | x \in A, y \geq f(x)\}$

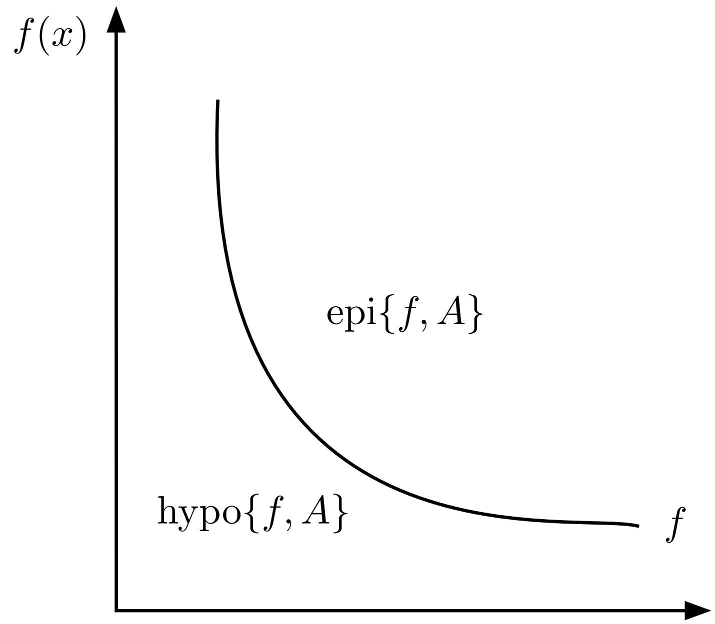
Theorem

(a) The function f is concave iff its hypograph is a convex set.

(b) The function f is convex iff its epigraph is a convex set.

Hypograph and epigraph of f - Illustration





Concave and convex functions - Properties

Theorem

- Let $f : A \subset \mathbf{R}^n \to \mathbf{R}$ be concave
- Let $g : \mathbf{R} \to \mathbf{R}$ be increasing and concave defined on an interval I containing $f(A)$.
- Then, $g[f(x)]$ is concave

Theorem

- Let $f, g : A \subset \mathbf{R}^n \to \mathbf{R}$ be concave functions
- Let $\alpha, \beta \in \mathbf{R}$
- Then, $h(x) = \alpha f(x) + \beta g(x)$ is concave

Theorem

- Let f be concave defined on an open set $A \in \mathbf{R}^n$.
- Then, f is continuous on A.

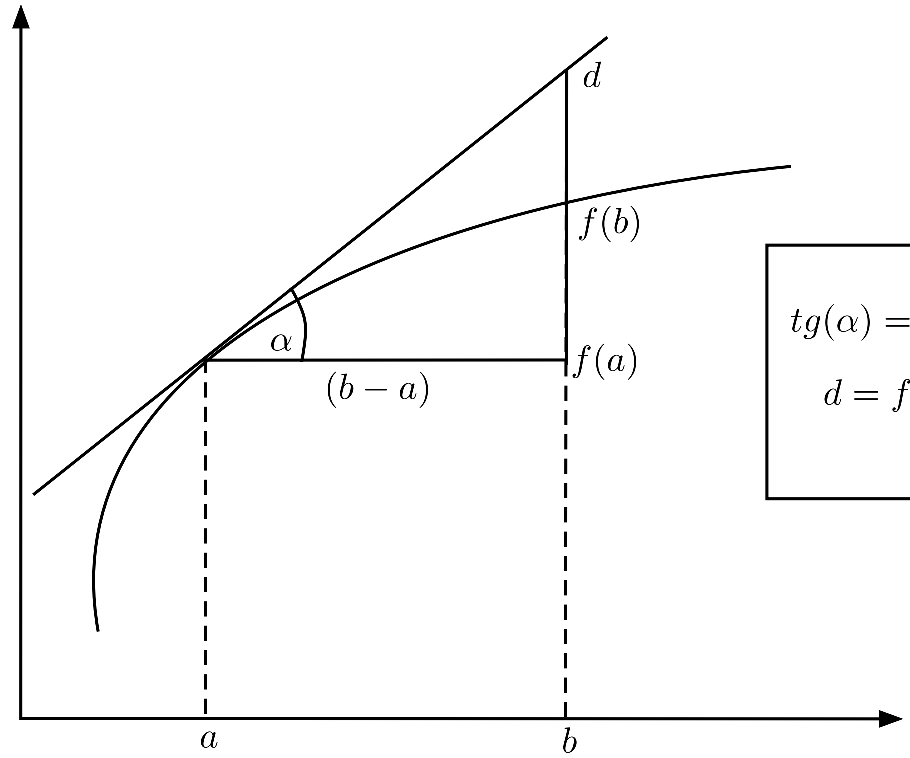
Concave and convex functions - Properties (2)

Theorem

- Let $f : A \subset \mathbf{R}^n \to \mathbf{R}$ be of class C^1 with A open and convex.
  (i) f is concave iff $\forall (a,b) \in A$, we have
  $f(b) \leq f(a) + Df(a)(b-a)$
  (ii) f is strictly concave iff $\forall (a,b) \in A$, we have
  $f(b) < f(a) + Df(a)(b-a)$

- The theorem says that f is (strictly) concave when the value of the function at b, $f(b)$ is smaller than or equal to the value of the linear approximation of f at a, evaluated at b.

Concave and convex functions - Properties (3)



$$tg(\alpha) = f'(a) = \frac{d - f(a)}{b - a}$$
$$d = f(a) + f'(a)(b - a)$$
$$d \geq f(b)$$

Concave and convex functions - Properties (4)

Theorem

- Let $f : A \subset \mathbf{R}^n \to \mathbf{R}$ be of class C^2 with A open and convex.
  (i) f is concave iff $\forall x \in A$, the Hessian matrix $D^2 f(x)$ is negative semidefinite.
  (ii) f is convex iff $\forall x \in A$, the Hessian matrix $D^2 f(x)$ is positive semidefinite.
  (iii) f is strictly concave iff $\forall x \in A$, the Hessian matrix $D^2 f(x)$ is negative definite.
  (iv) f is strictily convex iff $\forall x \in A$, the Hessian matrix $D^2 f(x)$ is posite definite.

Definiteness and (leading) principal minors

- M symmetric matrix $n \times n$;
- D_k leading principal minor order k.
- Δ_k principal minor order k
- A k^{th} order principal submatrix of M is a matrix that results from deleting *the same* $n-k$ rows and $n-k$ columns from M
- The leading principal submatrices of M are only those principal submatrices formed by deleting the last $n-k$ rows and $n-k$ columns.
- Theorem:
  - M is positive definite $\Leftrightarrow D_k > 0, \forall k$
  - M is negative definite $\Leftrightarrow sign D_k = sign(-1)^k, \forall k$
  - M is positive semidefinite $\Leftrightarrow \Delta_k \geq 0, \forall k$
  - M is negative semidefinite $\Leftrightarrow sign\Delta_k = 0$ or $(-1)^k, \forall k$

Definiteness and (leading) principal minors (2)

- Let $M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
- Leading principal minors: $\begin{vmatrix} a_{11} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$
- Second order principal minors:
  $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$
- First order principal minors: $\begin{vmatrix} a_{11} \end{vmatrix}, \begin{vmatrix} a_{22} \end{vmatrix}, \begin{vmatrix} a_{33} \end{vmatrix}$

Definiteness and (leading) principal minors (3)

- Consider $f(x, y, z)$ generating a Hessian matrix $M, 3 \times 3$
- f is strictly concave iff

$$\begin{vmatrix} a_{11} \end{vmatrix} < 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0;$$ [M neg def]

- f is concave iff
  - $\begin{vmatrix} a_{11} \end{vmatrix} \leq 0, \begin{vmatrix} a_{22} \end{vmatrix} \leq 0, \begin{vmatrix} a_{33} \end{vmatrix} \leq 0$
  - $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \geq 0, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \geq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0$
  - $\begin{vmatrix} M \end{vmatrix} \leq 0$

  [M neg semidef]

On Hessian definiteness

Example 1

- $M = \begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}$
- $D_1 = 1 > 0$
- $D_2 = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14 < 0$
- $M = \begin{vmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{vmatrix} = -109 < 0$
- Conclusion: M is indefinite

On Hessian definiteness (2)

Example 2

- $$M = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{pmatrix}$$
- $D_1 = 3 > 0$
- $$D_2 = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 > 0$$
- $$M = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{vmatrix} = 3 > 0$$
- Conclusion: M is positive definite

On Hessian definiteness (3)

Example 3

- $$M = \begin{pmatrix} -3 & -3 & 3 \\ 2 & 1 & 2 \\ 3 & -2 & 8 \end{pmatrix}$$
- $D_1 = -3 < 0$
- $$D_2 = \begin{vmatrix} -3 & -3 \\ 2 & 1 \end{vmatrix} = 3 > 0$$
- $$D_3 = \begin{vmatrix} -3 & -3 & 3 \\ 2 & 1 & 2 \\ 3 & -2 & 8 \end{vmatrix} = -27 < 0$$
- Conclusion: M is negative definite

On Hessian definiteness (4)

Example 4

- Assess whether $f(x,y) = x^2 - y^2 - xy$ is concave or convex
- $J(x,y) = \begin{pmatrix} 2x - y & -2y - x \end{pmatrix}$
- $H = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$
- $D_1 = 2 > 0$
- $D_2 = \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -5 < 0$
- H is indefinite $\to f(x,y)$ is neither concave nor convex.

On Hessian definiteness (5)

Example 5

- Assess whether $f(x, y) = 2x - y - x^2 + xy - y^2$ is concave or convex
- $J(x, y) = \begin{pmatrix} 2 - 2x + y & -1 + x - 2y \end{pmatrix}$
- $H = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$
- $D_1 = -2 < 0$
- $D_2 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 > 0$
- H is negative definite $\rightarrow f(x, y)$ is strictly concave.

On Hessian definiteness (6)

Example 6

- Assess whether $f(x,y) = 2x - y - x^2 + 2xy - y^2$ is concave or convex
- $J(x,y) = \begin{pmatrix} 2 - 2x + 2y & -1 + x - 2y \end{pmatrix}$
- $H = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$
- $D_1 = -2 < 0$
- $D_2 = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0$
- H is negative semidefinite $\rightarrow f(x,y)$ is concave.

On Hessian definiteness (7)

Example 7

- $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $D_1 = 1 > 0$
- $D_2 = 0$
- Conclusion: M is positive semi-definite

On Hessian definiteness (8)

Example 7

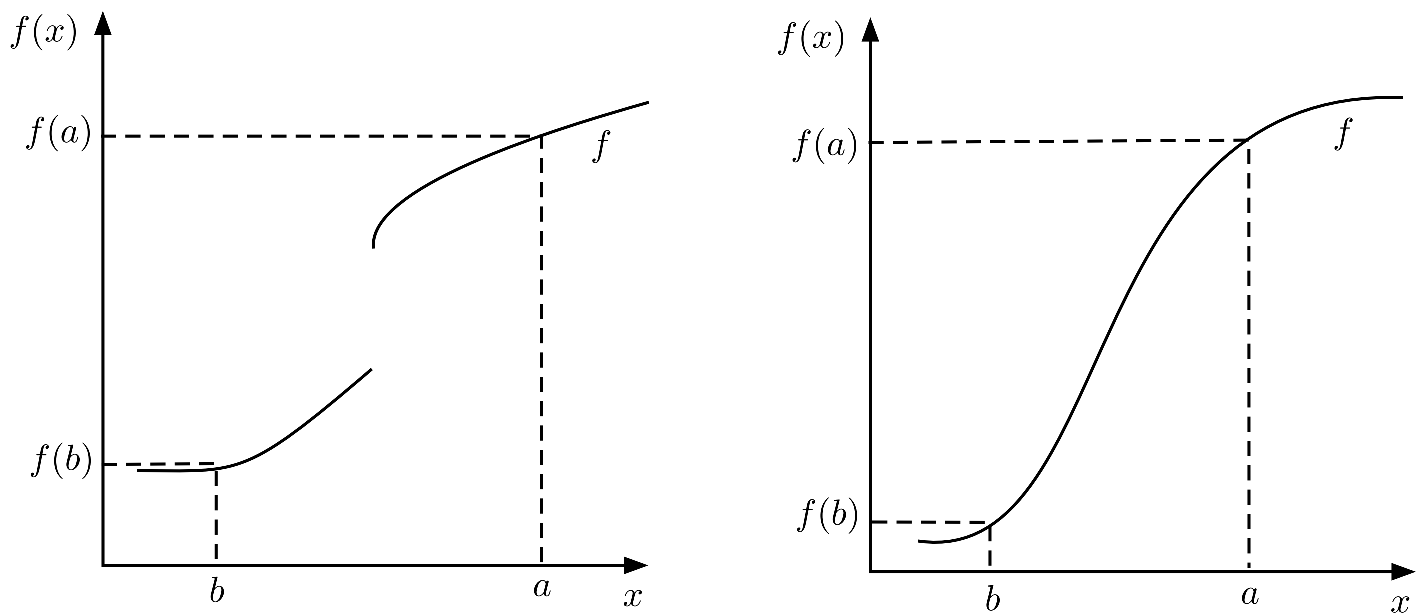
- Find the stationary ponits of $f(x, y) = x^3 - 3x^2 + y^3 - 3y^2$
- $J(x, y) = \begin{pmatrix} 3x^2 - 6x & 3y^2 - 6y \end{pmatrix}$
- Stationary points satisfy $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. These points are: $(0, 0), (2, 0), (0, 2), (2, 2)$
- Hessian matrix is $H = \begin{pmatrix} -6x - 6 & 0 \\ 0 & -6y - 6 \end{pmatrix}$
- Evaluate H at the stationary points

On Hessian definiteness (9)

- Consider $(0,0)$: $H = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$, $D_1 = -6,\ D_2 = 36$
  - negative definite: f strictly concave $\rightarrow (0,0)$ maximum.
- Consider $(2,0)$: $H = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}$, $D_1 = 6,\ D_2 = -36$
  - indefinite, thus a saddle point.
- Consider $(0,2)$: $H = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix}$, $D_1 = -6,\ D_2 = -36$
  - indefinite, thus a saddle point.
- Consider $(2,2)$: $H = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$, $D_1 = 6,\ D_2 = 36$
  - positive definite, f strictly convex $\rightarrow (2,2)$ minimum.

On quasi-concavity

- Intuition: Take any two points $(a, b) \in A$ and assume $f(a) \geq f(b)$.
- quasi-concavity requires that, as we move along the segment from the "low" point b to the "high" point a, the value of f never falls below $f(b)$.



f quasi-concave

On quasi-concavity (2)

- Recall: f is quasi-concave on A iff $\forall (a,b) \in A$ $\min\{f(a), f(b)\} \leq f((1-\lambda)a + \lambda b)$
- If f concave then f is quasi-concave
  - f concave: $f((1-\lambda)a + \lambda b) \geq (1-\lambda)f(a) + \lambda f(b)$
  - Also, $(1-\lambda)f(a) + \lambda f(b) \geq (1-\lambda)\min\{f(a), f(b)\} + \lambda \min\{f(a), f(b)\} = \min\{f(a), f(b)\}$
  - Thus, $\min\{f(a), f(b)\} \leq f((1-\lambda)a + \lambda b)$ and f is quasi-concave.

Local and global extreme points

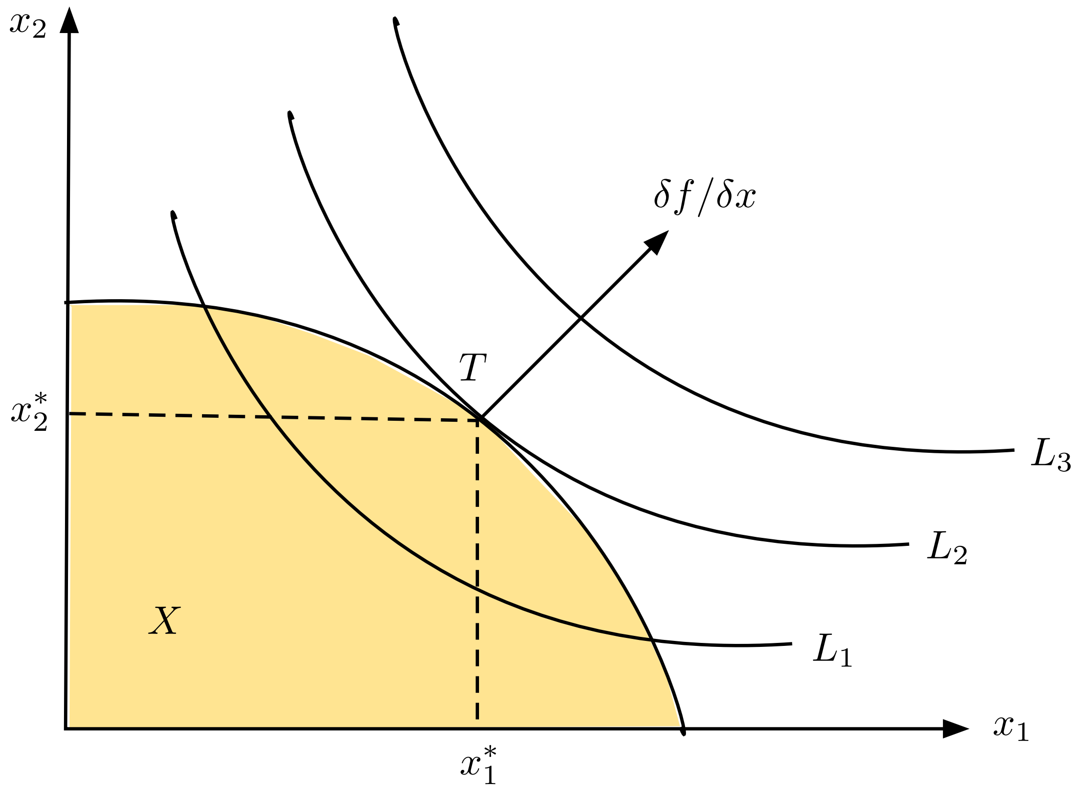
Theorem - Local-global

- Let $X \subset \mathbf{R}$ be convex and non-empty.
- Let $f : X \to \mathbf{R}$ be continuous and concave on X.
- Then, a local maximum is also global, and
- If f is strictly concave, then there is a unique maximum.

Local-global theorem - proof

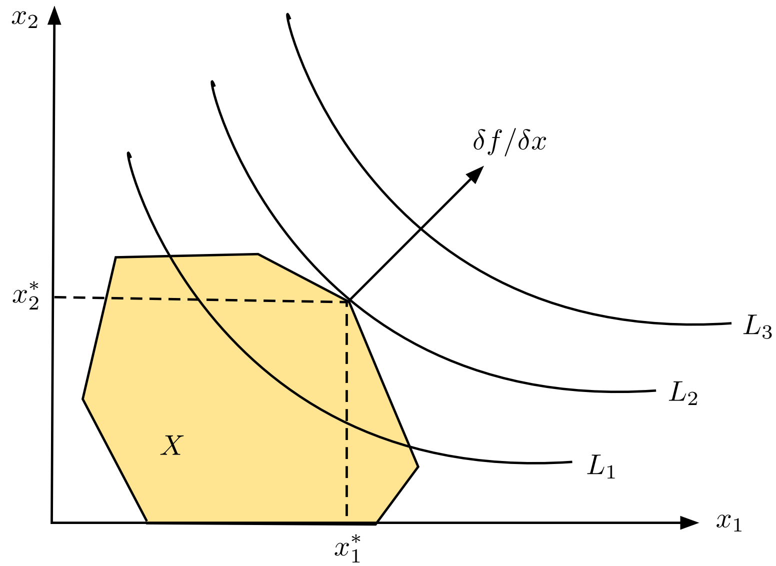
- Let $\hat{x}$ be a local maximum but not a global one.
- Then, $\exists r > 0$ such that $f(\hat{x}) \geq f(x), \forall x \in B(\hat{x}, r)$
- Since, $\hat{x}$ is not global max, $\exists y \in X$ s.t. $f(y) > f(\hat{x})$
- Since X is convex, $\forall \lambda \in (0,1),\ (1-\lambda)y + \lambda\hat{x} \in X$
  - pick $\lambda \approx 1$ so that $(1-\lambda)y + \lambda x \in B(\hat{x}, r)$
- By concavity of f, $f[(1-\lambda)y + \lambda\hat{x}] \geq (1-\lambda)f(y) + \lambda f(\hat{x})$
  - Since $f(y) > f(\hat{x})$ it follows $(1-\lambda)f(y) + \lambda f(\hat{x}) > f(\hat{x})$
- By construction, $(1-\lambda)y + \lambda x \in B(\hat{x}, r)$, implying $f(\hat{x}) \geq f[(1-\lambda)y + \lambda\hat{x}] > f(\hat{x})$. Thus, $f(\hat{x}) > f(\hat{x})$!!
- Accordingly, whenever $\hat{x}$ is a local max, it must also be a global one.

Geometry of Classical programming

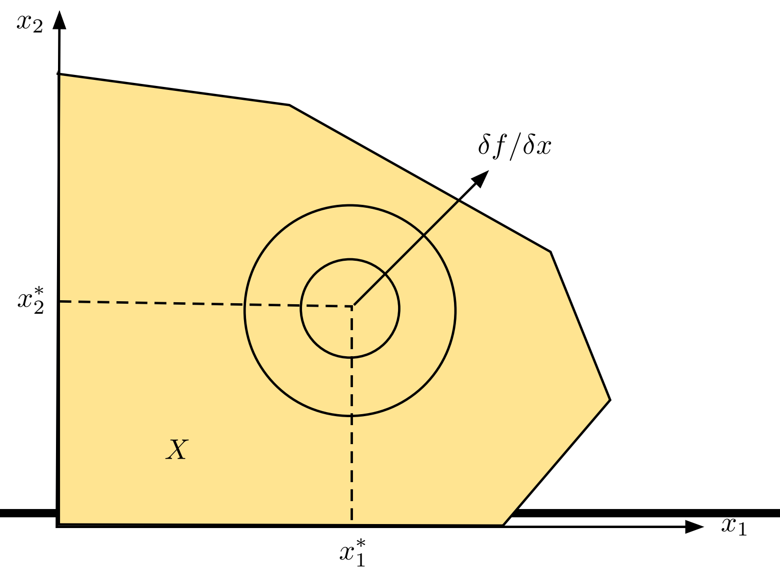


(a) Classic programming: tangency solution

Geometry of Non-linear programming

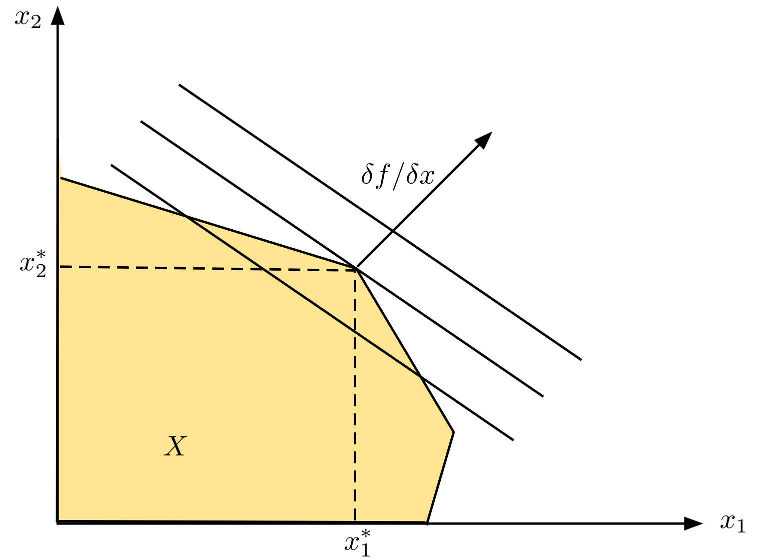


(a) Non-linear programming: corner solution

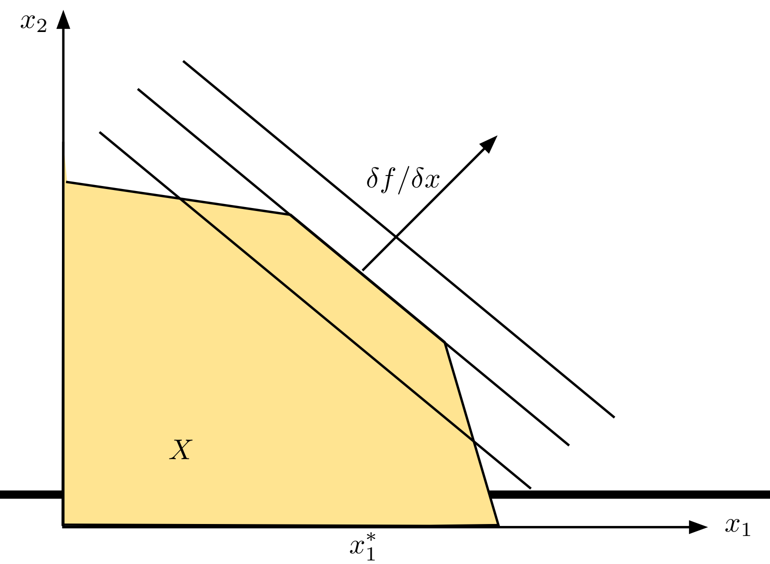


(a) Non-linear programming: interior solution

Geometry of Linear programming



(a) Linear programming: solution at a vertex



(b) Linear programming: continuum of solutions