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Lecture notes on Microeconomics

by

Łukasz Woźny

Warsaw School of Economics

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tel. 54 423 20 40
fax 54 423 20 80
e-mail: agencja.top@agencjatop.pl

Preface

These notes are prepared for the Microeconomic courses I teach at the Warsaw School of Economics. They are aimed to serve as a supplementary material for Microeconomic course at the introductory or intermediate level. The material covers canonical first level microeconomic topics including: consumer and producer choice, as well as competitive and monopolistic (partial) equilibrium analysis. If time allows (and it usually did during 15 meetings, hour and a half each) I also recommend to cover additional topics including: choice under uncertainty, introduction to non-cooperative games, selected issues from industrial organization or externalities (including analysis of public goods). Finally to introduce the reader to more advanced microeconomic topics I have prepared two short chapters on general equilibrium analysis as well as economics of asymmetric information. These are, however, only sketched here. The selection of material covered can depend also on major taught, and can vary between economics, finance or management. Each chapter includes a separate section with (subjectively selected) references to some important further readings. Finally, although material presented here is usually more than enough to cover during a standard one semester course, some important economic topics / disciplines are missing here (including public choice, mechanism design, cooperative game theory to mention just a few).

Clearly, the notes are far from being complete and cannot compensate for reading a full textbook on Microeconomics. One reason is that some (important) details are missing here. Firstly, whenever not restrictive to present the main argument I use standard tools from constrained optimization for differentiable objectives and constraining functions, hence ”non-smooth”/discrete case is not covered here. Secondly, as the exposition is mainly aimed to show the basic trade-offs but not solve all the problems, I only occasionally discuss the second order optimality conditions. Thirdly, when presenting some theorems or statements I miss their proofs but give a reference for such or sketch an argument when necessary. I tried to keep the exposition clear, though.

Writing these lectures I used Besanko and Braeutigam (2011), Varian (1992), Mas-Colell, Whinston, and Green (1995) textbooks and which I recommend for a more detailed treatment of topics covered here (for introductory, intermediate and advanced level respectively). I also recommend a textbook by Nicholson and Snyder (2012) that presents intuitively and exemplifies many concepts covered in these notes.

Finally, I want to thank Pawel Dziewulski for reading an early draft of these notes as well as the Department of Economics, University of Oxford for hosting during the writing of these notes.

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Contents

Chapter 1

Introduction to economic methods

A traditional definition of economics, advocated by Lionel Robbins, says that *Economics is the science which studies human behavior as a relationship between ends and scarce means which have alternative uses.* Resources are typically limited but needs need not be, hence economists analyze trade-offs how to allocate their constrained resources towards alternative uses. The typical questions that can be addressed within this framework are: what goods and services to produce, how to produce them efficiently, how to allocate goods and services among consumers, how to allocate gains from production/trade among consumers etc. These questions can be analyzed at **micro** or **macro** level. From perspective of economic theory this distinction is, to a large degree, irrelevant but for applied economics it is important, as microeconomics studies behavior of individuals (like consumers, producers, firms, managers etc.), while macroeconomics studies behavior of aggregate variables (like employment, gross domestic product, inflation etc.). Traditionally economic questions (or more precisely answers) are divided into **positive** and **normative**. Positive economics explains how economy works or predicts some future trends, while normative economics is more concerned in social welfare and policy recommendations.

Some more modern definitions of economics stress that economics deals with incentives, or as Steven Landsburg puts it: *Most of economics can be summarized in four words: people respond to incentives. The rest is commentary.* This means that the fundamental question in economics is the analysis of incentives that govern individual choices but also how to design or manipulate incentives so that responding individuals will behave in a desired way.

To a large extend economics is an operational science, i.e. economists try to solve real life/economy problems. This does not mean that economists do not use formal or theoretical tools. On the contrary much of economic research is based on abstract models. This requires a comment. Real life economic problems are typically very complicated and it is difficult to analyze them in their full complexity. For this reason economists create models that are supposed to represent the main / important trade-offs and problems of interest, but still abstract from other (non critical) elements. Hence, by construction models are not realistic and based on questionable assumptions. For this reason economists say that: *All models are unreal but some are useful*, stressing applicability of model's results. This argument has been taken to extreme by Milton Friedman in his "as if" methodological proposal. Friedman stressed that until model's results fits the real data we can say that individual or economy behaves "as if" it was generated by the model. In such case we say that model represents reality. This is indeed appropriate in some applications, but still one shall be careful, when going with this hypothesis too far. More on methodology of economics can be found in a book by Blaug (1992).

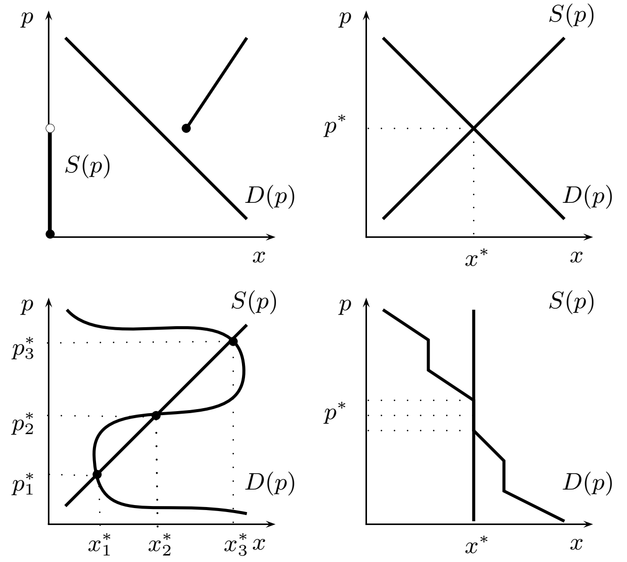
There are three typical methods economists use: constrained optimization, equilibrium analysis and comparative statics. We now briefly describe each of them.

Example of a **constrained optimization problem** is:

$$\begin{aligned}&\max_{x \in X} f(x, \theta),\\ &\text{s.t. } g(x, \theta) \leq m,\end{aligned}$$

where X is a **domain**, $f : X \times \Theta \to \mathbb{R}$ is an **objective function**, $g : X \times \Theta \to \mathbb{R}^n$ is a **constraint** function and m, θ are **parameters** (vectors in $\mathbb{R}^n$ and Θ). Variable x is called **endogenous** as it is chosen/determined within the problem, while m, θ are **exogenous** (parameters). Set $\{x \in X : g(x, \theta) \leq m\}$ is called a **feasible set**. A typical example of an objective is a profit of the company, where X, g represent technological constraints. Another example of an objective is a utility of the consumer, X his consumption set, while $g(x, \theta) \leq m$ represents her budget constraint. Finally, we can think of f as representing some social preferences, and the decision problem is to find the socially optimal outcome. Note that, these are all maximization problems, but economists sometimes analyze minimization problems, e.g. minimize costs of producing at least x units of output. All in all, constrained optimization is part of a decision theory and is one of the most typical economic tools. As we mentioned before, economists use constrained optimization problems for various reasons. They (i) solve 'real life' constrained optimization problems to find the best feasible solution; or (ii) assuming a family of optimization problems (for various θ) is given they estimate parameters θ from the observed data, such that solutions to the optimization problem are similar to the one observed in the real data. Finally, they also (iii) 'invent' constrained optimization problems, whose solutions coincide with observed choices. In (i) they often create a recommendation for a decision maker, forecast some economic variables or explain incentive and trade-offs that decision maker must be aware of; in

Figure 1.1: Examples of equilibrium analysis.



(ii) they provide information about parameters θ that can be used for other economic considerations (comparisons, forecasts), or provide interpretation of identified θ in behavioral terms; finally in (iii) economists must invent X, f, g so that the solutions to constrained optimization problem represent (usually uniquely) an observed real life data. This is linked to **revealed preference** argument, as decision maker reveals his objective and constraints via actual choices.

The second tool economists often use, is an **equilibrium analysis**. At this level we use two general notions of equilibrium: (i) competitive equilibrium of an economy in their partial or general incarnation and (ii) Nash equilibrium of a game. Both notions of equilibrium can be understood as a fixed point, i.e. a point $x^* \in X$ such that $f(x^*) = x^*$ for some $f : X \to X$. In the competitive (partial) equilibrium example, an equilibrium is a pair of price and quantity (p^*, x^*), such that supply equals demand $S(p^*) = x^* = D(p^*)$, which could be represented by a fixed point of some function f. Figure 1.1 indicates (for the competitive equilibrium) equilibria may be non-existent, unique, multiple or even form a continuum. A notion of general equilibrium is more complicated and we leave it to be defined later in chapter 9. Similarly in a game, where some players play against each other we would like to find a stable solution such that no player has an incentive to change its own decision assuming that other will also do not change theirs. In such case a (Nash) equilibrium is a fixed point of so called best response map f. Having established equilibrium existence economist conduct parameter estimation, forecasts or just explain observed phenomenas. Also revealed preference arguments are used. Finally, to mention, both supply and demand (or best responses in the game equilibrium context) are functions/correspondences derived from constrained optimization problems.

Finally for both types of tools: constrained optimization and equilibrium analysis economists conduct **comparative statics exercises**[1]. That is, they analyze how a solution of a constrained optimization problem or equilibrium changes, if some parameters θ of the primitives are changed. This includes forecasting or explaining possible reactions to changes in policy parameters or tastes, for example.

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[1]Historically the classic "first order" comparative statics was formalized by Samuelson (1947). See Milgrom and Shannon (1994), Topkis (1998), Amir (2005) or Quah (2007) for some modern developments in comparative statics.

Chapter 2

Consumer theory

In this chapter we will analyze consumer choices. This is important to understand both (i) the demand for goods and services that consumers consume as well as (ii) supply of endowments the consumers (initially) possess (e.g. capital or labor). Our approach to understand consumer choices could be summarized in the following points:

1. observe real data about the pairs of consumption bundles chosen and prices of goods and services in that bundle,
2. construct preferences that represent the observed choices under observed prices,
3. represent constructed preferences by some utility function,
4. analyze theoretically (for the purpose of forecasting or past behavior explanation) the choices of consumers with derived utility function under various incomes and market prices.

The following two sections will address all four points. At this level, however, we start from the 2nd one rather than the 1st, i.e. we assume that the preferences of consumers are given and then conduct the analysis of points number 3 and 4. At the end we will go back to the 1st point.

2.1 Preferences

Consider a consumer faced with choices from a given set X. This is a set of all consumption bundles consumer may think to consume and is called a **consumption set**. At this level we will assume that $X \subset \mathbb{R}_+^K$, where K stands for the number of goods. That is, we will consider consumption of nonnegative amounts of perfectly divisible goods, and assume that number of goods is finite.

One should note that K could be very large, though. Specifically, when differentiating goods economists use at least these four criteria:

- physical characteristics (clearly apple is different from orange, by its color, size, taste, flavor, etc.),
- location (clearly for a consumer located in Warsaw the goods delivered on Sahara desert give much less satisfaction, than on place),
- time (clearly the goods promised to be delivered the next quarter give lower satisfaction, than goods consumed today),
- state of the world (clearly an umbrella gives higher utility, when it is raining then not).

One could perhaps think of some more criteria that differentiate goods, and the key insight, when doing so could be reasoned from so called **law of one price**: goods which prices differ, should be regarded in principal as distinct. That is, consumers need a good reason to purchase similar goods at different prices. All in all, one can easily see that in reality the number of goods can be infinite. At this level we assume that K is finite[1], though, could be very large.

The consumer is assumed to have preferences denoted by $\succeq$ over bundles in X. If for $x, x' \in X$ we write $x' \succeq x$ we mean that x' is weakly preferred to x. That is, consumer ranks his satisfaction from consuming x' weakly higher, than from consuming x, assuming both are available at no costs. Specifically, preferences describe consumer wants, but not what she wants from bundles she can afford. In what follows we need the following assumptions:

1. **preferences are complete**, i.e. for any $x', x \in X$, $x' \succeq x$, or $x \succeq x'$. Hence x' could be preferred to x, or x could be preferred to x' or both, meaning that consumer is indifferent between x' and x. In such a case we write $x' \sim x$,
2. **preferences are transitive**, i.e. if $x'' \succeq x'$ and $x' \succeq x$, then we require that $x'' \succeq x$. This assumptions means that consumer makes consistent choices,
3. **weak monotonicity**, i.e. if $x' \geq x$, then[2] $x' \succeq x$. This assumption states that more is better: if bundle x' has weakly more units of goods than x, than x' must be preferred to x. However, if $x' \ngeq x$, i.e. bundle x' has more food but

[1] The analysis of consumer choice when the number of goods is infinite is important at the macroeconomic level, as economists analyze the dynamic economies with the infinite time horizon. Every period the same physical good is consumed / traded but as the time horizon is infinite the number of goods is infinite as well. For a discussion see Stokey, Lucas, and Prescott (1989) chapter 15 or Aliprantis, Brown, and Burkinshaw (1990).

[2] By $\geq$ in $\mathbb{R}^K$ we mean a standard order on $\mathbb{R}^K$, i.e. $x' \geq x$ if all coordinates of x' are $\geq$ the x.

less clothing than x, then without additional information we cannot say which bundle is preferred. Weak monotonicity is sometimes strengthen to **strong monotonicity**, i.e. if $x' \geq x$ and $x' \neq x$, then $x' \succ x$ meaning that, if x' has strictly more units of some good and weekly more of the others, then x' must be strictly preferred[3] to x.

Preferences that are complete and transitive are called **rational**.

Strict monotonicity can be restrictive as it means that goods are good, so it may not apply to garbage or pollution. But if one interprets goods as less garbage or less pollution, the strict monotonicity may be often assumed.

There are two other important assumptions concerning continuity of preferences and their (strict) convexity. **Continuity** is technical and we will not discuss it at this level. **Convexity** (resp. strict convexity) assumption means that if $x'', x', x \in X$ are such that $x'' \succeq x$, $x' \succeq x$ (and $x'' \neq x'$ resp.) then for any $t \in (0, 1)$ we have $tx'' + (1-t)x' \succeq x$ (resp. $tx'' + (1-t)x' \succ x$). Hence, convexity means that consumer prefers to mix extreme bundles, rather than consume one of them. Strict convexity is not always required nor satisfied.

In our discussion on preferences so far we have used the ordinal ranking. That is, we know how consumer ranks / prefers bundle x' to x, but we do not know how much more he likes x' to x. To analyze this we will switch to cardinal ranking. By cardinal ranking we mean a ranking that can tell us, that consumer prefers e.g. x' twice as much as x. For this reason and also because operations on ordinal rankings are cumbersome economists introduce the concept of a utility function. **Utility function** $u : X \to \mathbb{R}$ is a function that assigns numbers to every basket. We say that utility function u **represents** preferences $\succeq$, if for any $x', x \in X$ such that $x' \succeq x$ we have $u(x') \geq u(x)$. It can be shown[4] that, if preferences are complete, transitive and continuous then there exists a continuous utility function that represents them. Observe that the value of utility does not have any particular interpretation, as any monotone transformation of utility function u represents the same preferences[5].

Having such representation of preferences we can concentrate on the analysis of utility $u(x_1, x_2, \ldots, x_K)$, where $x = (x_1, \ldots, x_K)$ denotes a typical bundle in $X \subset \mathbb{R}^K$. We start by introducing some important concepts. First we define **marginal utility**. The marginal utility of good i is denoted by MU_i, defined by $MU_i = \frac{\triangle u}{\triangle x_i}$ and interpreted as a rate at which utility changes as the level of consumption of good i raises, holding consumption of other goods constant. For the differentiable utility function $MU_i = \frac{\partial u}{\partial x_i}$.

[3] By this we mean that $x' \succeq x$ and $x \not\succeq x'$.

[4] See Debreu (1964) theorem.

[5] Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function (monotone transformation). If utility function u represents preferences $\succeq$ for any $x', x \in X$, then whenever $x' \succeq x$, $u(x') \geq u(x)$. At the same time $f(u(x')) \geq f(u(x))$. Hence, $f \circ u$ is also a utility function representing $\succeq$.

Another important concept is an **indifference curve**. This is a set of all bundles in X with equal utility. That is, a set of all bundles that consumer is indifferent between (see figure 2.1).

Suppose now we increase consumption of good i, how much we have to reduce the amount of good j consumed to keep the utility constant? This is measured by a **marginal rate of substitution** of x_i for x_j. It is denoted by $MRS_{i,j}$ and calculated as $MRS_{i,j} = -\frac{dx_j}{dx_i}|_{u=const.}$. It can be shown[6] that for differentiable utility the following useful formula holds: $MRS_{i,j} = \frac{MU_i}{MU_j}$. Graphically marginal rate of substitution is a tangent to the indifference curve and hence serves as a local approximation of the rate at which consumer is willing to trade-off goods in his basket along the indifference curve. It can be shown that marginal rate of substitution does not depend of utility representing particular preferences.

Let us now summarize the few basic properties of rational, strongly monotone preferences:

- When MU_i is positive for all goods i, then indifference curves have a negative slope. Why?
- Indifference curves for two different utility levels cannot intersect. Why?
- Every bundle lies on only one indifference curve. Why?
- Indifference curves are not thick. Why?

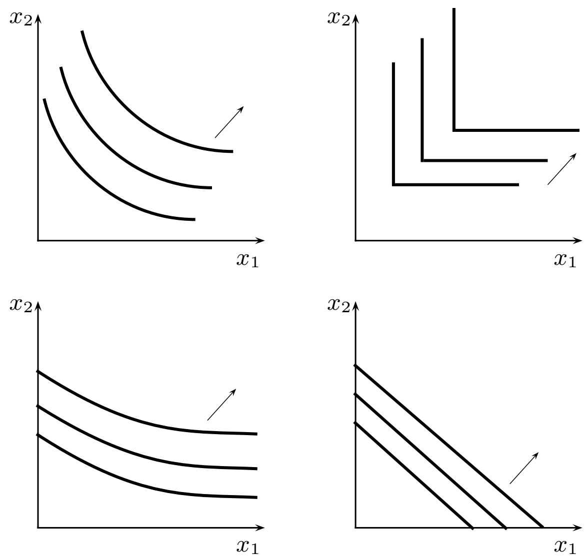
Now in a few examples for $K = 2$, we discuss important special cases of preferences and analyze their properties using concepts defined at the moment.

Example 2.1 (Cobb-Douglas) *A typical example of a utility function is a Cobb-Douglas function* $u(x_1, x_2) = x_1^\alpha x_2^\beta$ *for* $\alpha, \beta \geq 0$. *The marginal utilities are* $MU_1 = \alpha x_1^{\alpha-1} x_2^\beta$ *and* $MU_2 = \beta x_1^\alpha x_2^{\beta-1}$. *Observe that* MU_1 *is positive but* $x \to MU_1$ *is decreasing for* $\alpha < 1$. *The last property is called* <u>*diminishing marginal utility*</u> *and denotes preferences for which each next unit of good* 1 *gives less and less utility for a consumer. The marginal rate of substitution:* $MRS_{1,2} = \frac{\alpha x_2}{\beta x_1}$ *and again* $x_1 \to MRS_{1,2}$ *is decreasing due to convexity of preferences.*

Example 2.2 (Prefect substitutes) *Consider* $u(x_1, x_2) = \alpha x_1 + \beta x_2$. *The* $MU_1 = \alpha$, $MU_2 = \beta$ *and are constant rather then decreasing. The* $MRS_{1,2} = \frac{\alpha}{\beta}$. *This means that consumer has a constant rate at which she is willing to exchange goods at the indifference curve, as both goods are perfectly substitutable.*

[6] By construction changes dx_i and dx_j satisfy $MU_i dx_i + MU_j dy_j = 0$. Hence $\frac{dx_j}{dx_i} = -\frac{MU_i}{MU_j}$.

Figure 2.1: Examples of indifference curves for convex preferences: perfect complements (right, top panel); perfect substitutes (right, bottom panel); quasilinear (left, bottom panel).



Example 2.3 (Perfect complements) *Consider $u(x_1, x_2) = \min\{\alpha x_1, \beta x_2\}$. This means that consumer's utility is always limited by the minimal of the two numbers: consumption of αx_1 or βx_2. It means that consumer needs to consume these goods in fixed proportion $\frac{\beta}{\alpha}$. Observe that such indifference curve is not differentiable at any point (x_1, x_2), such that $\alpha x_1 = \beta x_2$.*

Example 2.4 (Quasilinear preferences) *Let $u(x_1, x_2) = v(x_1) + x_2$. Such preferences are called quasi-linear in x_2, as the utility level depends linearly in x_2, but not necessarily in x_1. The $MU_2 = 1$ and $MRS_{2,1} = \frac{1}{v'(x_1)}$ and is constant in x_2.*

Example 2.5 (Homothetic preferences) *An important class of preferences are so called homothetic preferences. Formally homothetic preferences are defined by the following implication: if $u(x) = u(x')$ for two bundles $x, x' \in X$, then $u(tx) = u(tx')$ for any $t \in \mathbb{R}_+$. It means that, if the consumer is indifferent between two baskets, then he will be still indifferent, if we multiple two baskets by a scalar.*

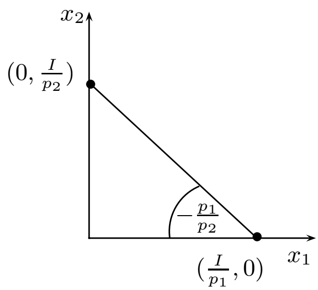
The next figure 2.1 exemplifies indifference curves representing preferences discussed so far.

2.2 Choice

Here we will analyze a consumer's choice. Let $X \subset \mathbb{R}^K_+$ and denote a typical bundle $x = (x_1, x_2, \ldots, x_K)$. We start by defining a budget set. Given income I and prices $p = (p_1, p_2, \ldots, p_K)$ of all goods, a budget set is a subset of X that consumer can afford. A typical example of a budget set is $B(p, I) = \{(x_1, x_2, \ldots, x_K) \in X \subset \mathbb{R}^K_+ : \sum_{i=1}^K x_i p_i \leq I\}$. Note, that for any positive prices p, budget set is convex, which will play an important role in our analysis[7]. The relation $\sum_{i=1}^K x_i p_i = I$ defines a budget constraint. If one considers nonlinear prices, e.g. some price discounts, if consumer buys at least a number of particular goods, then the budget set could be given by more than one inequality and need not be convex.

Example 2.6 *For a two goods case x_1, x_2 the budget constraint can be written as $x_1 p_1 + x_2 p_2 \leq I$ and the slope of the budget line is given by $-\frac{p_1}{p_2}$. An increase in income shifts the budget parallelly, further from the origin. An increase in the price of a particular good changes the slope of budget line $-\frac{p_1}{p_2}$ accordingly. The slope of the budget line gives the ratio at which one can exchange good 1 for 2 at the market. This is illustrated in figure 2.2.*

Figure 2.2: A typical budget set.



Having that, the problem of a consumer is to maximize his utility subject to budget constraint:

$$\max_{(x_1, x_2, \ldots, x_K) \in X} u(x_1, x_2, \ldots, x_K), \tag{2.1}$$
$$\text{st.} \sum_{i=1}^K x_i p_i \leq I.$$

[7]Set A is convex, if for any two elements in $a, a' \in A$, and a scalar $t \in (0, 1)$, $ta + (1-t)a' \in A$. Now take any $x, x' \in B(p, I)$ and a scalar $t \in (0, 1)$. Observe that $\sum_{i=1}^K [tx_i + (1-t)x'_i]p_i = t\sum_{i=1}^K x_i p_i + (1-t)\sum_{i=1}^K x'_i p_i \leq tI + (1-t)I = I$. Hence $tx + (1-t)x' \in B(p, I)$.

We now analyze solutions to such problem.

First, one needs conditions guaranteeing that such a solution exists. For that, it is enough that u is continuous and prices positive. Additionally if preferences are strictly convex (i.e. utility function is strictly quasi-concave[8]) the solution is unique. Second, observe that solution to the utility maximization problem does not depend on particular u representing preferences. Third, if we multiply the income and prices by the same constant the budget constraint and hence solution remains unaffected. Hence we may also normalize prices such that $p_1 = 1$. And finally, for strongly monotone preferences a bundle that costs strictly less then income is never optimal, hence we can alternatively express the inequality constraint as equality: $\sum_{i=1}^{K} x_i p_i = I$.

Formally the solution to problem (2.1) is denoted by $x^*(p, I)$. If $x^*(p, I)$ is unique we call x^* a (Marshallian) demand function. The maximal utility, that can be obtained, is denoted by $v(p, I) = u(x^*(p, I))$ and called a value or **indirect utility function**. The indirect utility function can be used to construct direct money-metric utilities[9].

If the utility function is differentiable, than any optimal, interior bundle must satisfy[10]:

$$(\forall i, j) \quad \frac{MU_i}{p_i} = \frac{MU_j}{p_j}. \tag{2.2}$$

This says that at the optimal, interior bundle the marginal rate of substitution must be equal to the ratio of prices for any two goods. The condition is intuitive as it requires that the rate of which consumer is willing to exchange any two goods is equal to the rate at which market does so.

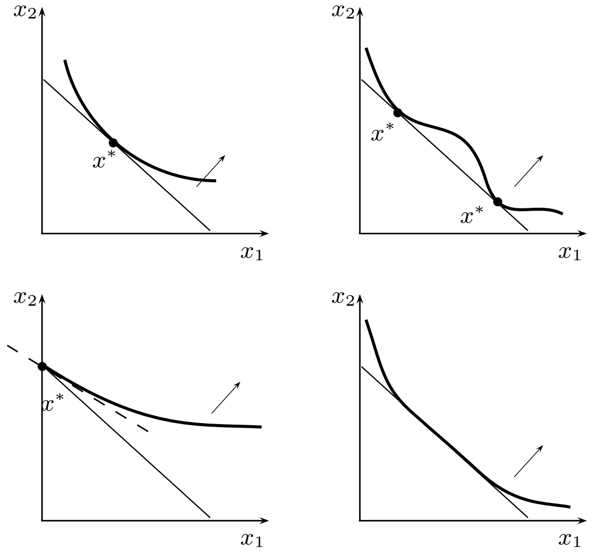
This condition is also sufficient if the second-order conditions are satisfied. Can you state and interpret them? The condition (2.2) is necessary for interior solutions but not for corner solutions. By corner solution we mean optimizing bundles with $x_j = 0$ for at least one j. To find the global solution (interior or corner) one have to compare utility levels at the interior solution using (2.2) and compare it to the

[8]To see that observe the following. Let $V = \{x \in X : x \succeq y\}$ be the preferred set. Then by convexity of preferences we must have that for any $t \in (0, 1)$ and any $x, x' \in V$ we have $tx + (1-t)x' \in V$. As a result convex preferences have convex sets V for any y. Now for a utility u representing $\succeq$ we must have $V = \{x \in X : u(x) \geq u(y)\}$. By definition, for any quasi-concave functions its upper contour set V is convex.

[9]By this we mean a utility defined over the consumption set of pairs: price and income. Such function can be interpreted as a utility of 'consuming' pairs p, I directly.

[10]Consider a corresponding Lagrange function $L(x, \lambda) = u(x) + \lambda(I - \sum_i x_i p_i)$. In our case the Slater constraints qualification is satisfied and solution to $\max_x L(x, \lambda^*)$ equals the solution of problem (2.1). Alternatively a pair (λ^*, x^*) solves $\min_{\lambda \in \mathbb{R}_+} \max_{x \in X} L(x, \lambda)$. Calculating derivatives with respect to x_i and equating them to zero (for interior solution) $\frac{\partial u}{\partial x_i}(x^*) - \lambda^* p_i = 0$ we obtain the desired equality. The envelope theorem assures that under certain conditions the value of Lagrange multiplier $\lambda^* = \frac{\partial v}{\partial I}(p, I)$ and can be interpreted as the marginal utility of income. More on Lagrange or Kuhn-Tucker methods for convex, constrained optimization problems can be found in Rockafellar (1997) or Clarke (1983), Rockafellar (1981) for nonsmooth case.

Figure 2.3: Consumer choice. Unique solution (left panels) and multiple solutions (right panels).



utility levels for all possible corner solutions. Figure 2.3 presents example of a unique interior, and corner solutions, as well as multiple solutions to consumer maximization problem.

Remark 2.1 *The utility maximization problem is* ***dual*** *to the following expenditure minimization problem:*

$$e(p, v) = \min_{x \in X} \sum_i x_i p_i,$$
$$st.\ u(x) \geq v,$$

that is a problem of minimizing the total cost of obtaining the utility level v*. The solution to this problem is denoted by* $h^*(p, v)$ *and called the Hicksian demand (function). Clearly* $e(p, u^*) = I$*, where* $u^* = v(p, I)$*; also* $h^*(p, v(p, I)) = x^*(p, I)$ *and* $h^*(p, v) = x^*(p, e(p, v))$*.*

Example 2.7 (Derive demand for Cobb-Douglas utility) *We consider a utility function from two goods* $u(x_1, x_2) = x_1^\alpha x_2^\beta$ *as before. Let income* $I > 0$ *and prices* $p_1, p_2 > 0$ *by given. Then the optimal bundle* (x_1^*, x_2^*) *satisfies* $\frac{p_1}{p_2} = MRS_{1,2} = \frac{MU_1}{MU_2} = \frac{\alpha x_2^*}{\beta x_1^*}$ *as well as the budget constraint* $x_1^* p_1 + x_2^* p_2 = I$*. Solving this system of two equalities we obtain that the optimal bundle is given by* $(x_1^*, x_2^*) = (\frac{\alpha}{\alpha+\beta}\frac{I}{p_1}, \frac{\beta}{\alpha+\beta}\frac{I}{p_2})$*. Observe that demand increases with income and decreases with price of a good.*

Example 2.8 (Derive demand for CES utility) *Consider a CES utility function* $u(x_1, x_2) = [x_1^\rho + x_2^\rho]^{\frac{1}{\rho}}$, *where* ρ *is a parameter. Then the optimal bundle* (x_1^*, x_2^*) *satisfies* $\frac{p_1}{p_2} = MRS_{1,2} = \frac{MU_1}{MU_2} = \frac{\rho x_1^{\rho-1}}{\rho x_2^{\rho-1}}$ *as well as the budget constraint* $x_1^* p_1 + x_2^* p_2 = I$. *Solving this system of two equalities we obtain that the optimal* x_1^* *is given by* $\frac{p_1^{\frac{1}{\rho-1}} I}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}$ *and similarly for* x_2^*.

Example 2.9 (Labor supply) *Consider a household choosing a consumption level* c *and a number of hours worked* l. *The time is normalized to one and the hours spend not working (leisure) are denote by* $h = 1 - l$. *Consumer has wealth* $T \geq 0$, *and the hourly wage is* w. *If the price of a consumption good* p *is normalized to 1, then the corresponding budget constraint is given by:* $c \leq wl + T$. *Consider preferences over consumption and leisure given by* $u(c, 1-l)$. *The optimal (consumption, leisure) basket satisfies* $\frac{u_2'(wl^*+T, 1-l^*)}{u_1'(wl^*+T, 1-l^*)} = w$ *and says that the marginal rate of substitution between leisure and consumption must be equal to the ratio of prices* $\frac{w}{1}$. *Our analysis derives the demand function for consumption and leisure but also individual supply of labor hours.*

Example 2.10 (Borrowing/lending) *Consider a household choosing a consumption level in the first period* c_1 *and the second period* c_2. *Let preferences be given by* $u(c_1) + \beta u(c_2)$, *where* $\beta \in (0, 1)$ *is a discount factor. The consumer earns* w_1 *in the first period and* w_2 *in the second. Consumer can spend income* w_1 *on consumption* c_1 *or savings* s *(that could be positive or negative). Savings give interest* r *the next period. The budget constraints for the consumer are:* $p_1 c_1 + s \leq w_1$ *and* $p_2 c_2 \leq w_2 + s(1+r)$. *Rearranging:* $p_1 c_1 + \frac{1}{1+r} p_2 c_2 \leq w_1 + \frac{1}{1+r} w_2$ *which states that the present value of consumption stream cannot exceed the present value of income. The interior, optimal consumption basket satisfies:* $\frac{u'(c_1^*)}{\beta u'(c_2^*)} = \frac{p_1}{\frac{1}{1+r} p_2}$. *Interpreting the marginal rate of substitution between the periods must be equal to the ratio of prices between the periods. Observe that our analysis derives demand for consumption in both periods but also demand/supply of borrowing/lending.*

The demand function derived from the consumer maximization problem can have various properties as a function of income and price. We can now use them to present a bacis categorization of goods.

- If demand is increasing as a function of income, we say that good is **normal**.
- If demand is decreasing as a function of income, we say that good is **inferior**.
- If income elasticity[11] of demand exceeds one, than the good is called **luxury**.

[11] For a differentiable function $f : \mathbb{R} \to \mathbb{R}$ we define its elasticity at point x by $\varepsilon_x^f := \frac{f'(x)x}{f(x)}$. A

A typical example of a normal good is a case of unitary income elasticity of demand. This holds for homothetic preferences. In such a case the consumer always spends the same fraction of income on a given good. Similarly we can consider a demand as a function of price. Typically demand should be a decreasing function of price but in the case of a Giffen (e.g. the case of demand for potato in Ireland during the Great Famine) or Veblen (snob effect) good, where demand is increasing with price, is also possible. For some new developments in comparative statics in consumer theory see Quah (2003), Antoniadou (2006) and Mirman and Ruble (2008) along with references contained within.

It happens that the demand change, as a result of a price change, can be decomposed into two effects: **substitution** and **income**. If the price decreases, the former effect means that the consumer can now substitute his consumption towards relatively cheaper goods. The latter one means that as the price of some goods has decreased the consumer can purchase more as he has relatively more money. Formally speaking, consider the demand for good x_j, then the following Slutsky equation summarize such decomposition:

$$\triangle x_j \approx \frac{\partial x_j(p,I)}{\partial p_i}\triangle p_i = \frac{\partial h_j(p,v(p,I))}{\partial p_i}\triangle p_i - \frac{\partial x_j(p,I)}{\partial I}x_i(p,I)\triangle p_i.$$

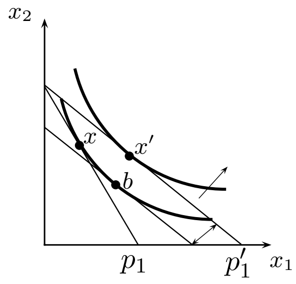
Its derivation is omitted. The first term in the Slutsky equation is a substitution effect (changes to the Hicksian demand keeping utility constant), while the latter is the income effect. The following figure 2.4 illustrates this construction.

Another way to measure, how the changes in prices influence demand and utility in monetary terms, are **compensating** and **equivalent variation**. The compensating variation uses the new prices as a base, and answers what income change in current prices can compensate (i.e. gives the same utility) for the price change. Similarly equivalent variation uses the old prices as a base, and answers what income change in current prices is equivalent (i.e. gives the same utility) to the price change.

The analysis we did so far assumed that we know prices and consumer preferences or its representation by a utility function. As mentioned in the introduction to this chapter the analysis should start, however, with the observed choices. The **revealed preference** analysis allows us to construct the preferences that **rationalize** observed

price elasticity of demand function d is denoted by $\varepsilon_p^d = \frac{\frac{\triangle d}{d}}{\frac{\triangle p}{p}}$, where $\triangle d = d_2 - d_1$ and similarly, $\triangle p = p_2 - p_1$. For differentiable demand function d, $\varepsilon_p^d = \frac{d'(p)p}{d(p)}$. Hence price elasticity of demand measures % change in demand for a one % change in price. Similarly one introduces a concept of price elasticity of supply $\varepsilon_p^s = \frac{\frac{\triangle s}{s}}{\frac{\triangle p}{p}}$, where s is a supply function. Other useful elasticities can also be defined including: income elasticity of demand, measuring relative changes in demand to relative changes in income; or cross-price elasticity of demand, measuring relative changes in demand for good 1 to relative changes of price of good 2.

Figure 2.4: Hicksian decomposition into substitution and income effects. Price decrease from p_1 to p_1' and demand changes from x to x'. Change $b-x$ reflects substitution effect (utility is the same for both baskets), while $x'-b$ reflects income effect (parallel shift in the budget line).



choices. The idea behind revealed preferences is simple. Suppose we observe consumer's choice of bundle x under prices p. Then for any other bundle x' whose price $p \cdot x' \leq p \cdot x$ it must be that $x' \preceq x$, as the consumer could afford bundle x' but did not choose it. In such case we say that x is directly weakly revealed preferred to x'. Similarly for any bundle x'' whose price was $p \cdot x'' < p \cdot x$ we must have $x'' \prec x$, and we say that x is directly strictly revealed preferred to x''. Similarly we consider indirect revealed preferences via transitivity. Having enough observation, and assuming consumer preferences do not change, we can construct the utility function that consumer has revealed to us by its choices.

Formally, this can be summarized in the following axiom (GARP): if basket x is (directly or indirectly) weakly revealed preferred to x', then x' cannot be (directly or indirectly) strictly revealed preferred to x. Having that we can state the celebrated Afriat's theorem.

Theorem 2.1 (Afriat (1967)) *The set of choices $(x_t, p_t)_{t=0}^{T}$ satisfies GARP if and only if there exists a continuous, monotone, concave utility function, that rationalize these choice (as outcomes from utility maximization).*

2.3 Demand and consumer surplus

The demand derived for each individual consumer should be aggregated to obtain the market demand for a particular good. This is done by summing n individual demands $\{x_i^*(\cdot)\}_{i=1}^{n}$ and denoted by: $D(p, I_1, \ldots, I_n) = \sum_{i=1}^{n} x_i^*(p, I_i)$. Hence, essentially the aggregate demand function inherits all the properties of the individual demand functions. However, if the number of consumers is large and they are sufficiently diversified, then aggregate demand function may be continuous even if the individual demands are not[12]. Apart from continuity (and homogeneity) the aggregate demand function will not possess other nice properties unless all the individual demands do.

The interesting case is that, under certain conditions, the aggregate demand function looks "as if" it was derived from the individual, **representative consumer**. The necessary and sufficient conditions for these (see Gorman, 1953) are that the indirect utility function takes the form $v_i(p, I_i) = a_i(p) + b(p)I_i$. Then the aggregate indirect utility function is simply $V(p, \sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} a_i(p) + b(p) \sum_{i=1}^{n} I_i$.

The final topic considers consumer welfare. The classic tool to measure it is a **consumer's surplus**. If the demand for a given good is as a function of price is given by $D(p)$, then the consumer surplus is simply $CS = \int_{p_0}^{p_1} D(p)dp$. It happens that if the utility function is quasilinear the CS is the exact measure of consumer welfare. In such a case the compensating and equivalent variation are equal to the change in consumers surplus.

We finish this chapter with references to Deaton and Muellbauer (1980) and Deaton (1992) with serious extensions and applications of the basic models treated here.

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[12] We refer the reader to the book of Trockel (1984) for formal argument.

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Chapter 3

Producer theory

In this chapter we analyze producer's behavior. This is important to understand (i) demand for input factors (like capital or labor), (ii) firm's cost functions, as well as (iii) technological progress. We start by introducing a standard way of describing technology using production function, discuss important aspects of technology and then define firm's cost and study a cost minimization problem. All of that will be necessary to analyze firm's profit maximization in the following chapters.

3.1 Technology and output

Technology transforms **inputs** (or production factors) into **outputs**. Examples of inputs include: labor services, physical capital, land, raw materials or intermediate goods, etc. Inputs are goods, so again we shall speak about specific inputs, differentiated not only with respect to physical characteristics, but also location, time or state of the world. Production usually takes time, but if one differentiates goods according to time, our description would be general enough to incorporate timing implicitly. Typically output and inputs are multidimensional: to produce a given good one usually needs many inputs and similarly usually one produces many outputs at the same time (e.g. energy and pollution). At this level we will focus on the examples with a single output and two inputs. Technology is typically summarized using a **production function** that transforms inputs into a maximal level of output that can be produced from a given combination of inputs. An example of a production function is given by f with values:

$$q = f(k, l),$$

where k stands for capital, l for labor and q for output. The set of inputs and outputs that are possible to produce using technology f is called a **production possibilities**

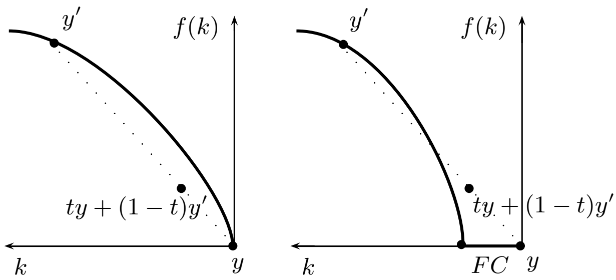
set and given by: $Y = \{(q, -k, -l,) : q \leq f(k,l), k \geq, l \geq 0\}$. The minus signs are supposed to capture that k, l are (net) inputs while q is an (net) output. The term net refers to a situation if some factor is used to produce a.o. itself (e.g. power plant uses usually a.o. electric energy to produce electric energy). Technology can change over time due to e.g. technological progress. For the time being we assume that technology is constant but later we will shortly discuss technological progress.

We now introduce some important concepts of average and marginal productivity. **Average productivity** of, say labor, is simply $AP_l = \frac{f(k,l)}{l}$, while **marginal productivity** $MP_l = \frac{\triangle f(k,l)}{\triangle l} = \frac{f(k,l+\delta)-f(k,l)}{(l+\delta)-l} = \frac{f(k,l+\delta)-f(k,l)}{\delta}$ and measure productivity of additional δ units of labor. Average productivity is hence global (i.e. depends only on the amount of inputs used), while marginal is a local measure (depends on the amount, as well as the change δ in the use of inputs). Again, for differentiable production function $MP_l = \frac{\partial f}{\partial l}$.

Typical assumptions on the production function or production possibilities set include:

- monotonicity: f is a nondecreasing function of inputs,
- convexity: f is quasiconcave[1] or equivalently, Y is convex,
- regularity: $V(q) = \{(k,l) : q \leq f(k,l)\}$ is closed and nonempty for all q.

Figure 3.1: Convex (left panel) and nonconvex technology set (right panel).



Monotonicity implies that more inputs allow to produce more outputs. This is true, if free disposal of inputs is possible. The regularity condition is technical and innocent in most applications. The strongest assumption concerns convexity. It means that, if any two combinations of production are possible, then the mixed combination is also possible. It is restrictive as it e.g. rules out technologies with fixed costs and $0 = f(0,0)$ feasible not mentioning concave production sets. This is illustrated in

[1] For convex X, we say a function $f : X \to \mathbb{R}$ is quasiconcave iff $(\forall x_1, x_2 \in X)$ and $\forall \alpha \in [0,1]$ we have $f(\alpha x_1 + (1-\alpha)x_2) \geq \min\{f(x_1), f(x_2)\}$.

figure 3.1 for the example of single input concave production function f without and with fixed costs (FC). We will sometimes dispense with convexity assumption in our analysis.

The set of factors that allows to produce exactly q is called an **isoquant**. Isoquant is supposed to show that level q is possible to be produced by different combinations of inputs, e.g. substituting capital for more labor or vice versa. The measure of such substitability is called **marginal rate of technical substitution** and denoted by $MRTS_{k,l} = -\frac{dl}{dk}|_{q=const.}$. Interpreting, $MRTS_{k,l}$ says: by how much labor input should be increased if capital is to be decreased by a unit to keep the level of production constant. If isoquant curve is differentiable the MRTS is simply the tangent to the isoquant curve at a given point, and hence a local approximation of trade-off (or substitution) between inputs on the isoquant. It happens that, if the production function is differentiable, then[2] $MRTS_{k,l} = \frac{MP_k}{MP_l}$.

The MRTS measure the slope of the isoquant, while the **elasticity of substitution** measures the curvature of the isoquant. Specifically, elasticity of substitution measures the percentage change in the factor ratio over percentage change of MRTS along the isoquant:

$$ES = \frac{\frac{\triangle k/l}{k/l}}{\frac{\triangle MRST_{l,k}}{MRST_{l,k}}}.$$

Typically ES is calculated using the so called logarithmic derivative: $ES = \frac{d \ln k/l}{d \ln |MRTS|}$.

Finally we introduce an important (global) concept of **returns to scale**. We say that technology exhibits increasing (resp. decreasing, constant) returns to scale if

$$(\forall k > 0, l > 0)(\forall A > 1) \quad f(Ak, Al) > Af(k, l) \quad (\text{resp.} \;\; <, =).$$

The increasing returns to scale assumption states that if we increase proportionally all the inputs in the production process than we could produce more than a proportional increase in output. That is, a characteristic of a production process for which a larger production is more efficient. Few typical sources of increasing returns to scale include:

- spreading fixed costs. Clearly the higher inputs the larger output to spread the fixed costs,
- indivisibilities in a production process,
- physical characteristics of a production process, e.g. the cube-square rule,
- risk sharing. Larger firm can spread small risks of a production more efficiently,

[2] Recall the definition of the marginal rate of substitution, presented in chapter 2.1. Observe that the two definition are very much alike. In fact, this observation will be useful, when discussing the theory of general equilibrium.

- using large scale technologies, i.e. although the single technology may posses constant or decreasing returns to scale at certain production levels, if the scale increases the company may switch to more efficient large scale technologies and actual production data would suggest an increasing returns to scale.
- few other including: economies of scale in purchases (higher discounts for higher orders) or marketing (higher hit ratio of marketing campaigns of larger firms).

Capital intensive technologies usually have increasing returns, while labor intensive constant or decreasing returns. Clearly increasing returns to scale violate convexity assumption. Can you verify that?

Replication argument would suggest that, in the worst case scenario the company may e.g. double its production by building a second factory aside, so the constant or increasing returns to scale assumption should be natural. Put differently, we shall not observe decreasing returns to scale in reality. However, there are examples of decreasing returns to scale technologies. The reason is simple, the returns to scale analysis assumes that all factors can be multiplied proportionally, however, sometimes it is not possible as some factors may be constrained or even fixed (e.g. natural resources available in a give region, or specific job supply in an area, or even communication possibilities, etc.). So typically, when we speak of decreasing returns one should think of **fixed** (production) **factors**.

Some factors may be fixed in the **short run** but become flexible in the **long run**. Specifically, we talk about a short run technology / production function, if one of the factors (like capital) is fixed: $q = f(\bar{k}, l)$, where $\bar{k}$ is some predefined constant.

Efficiency requires that for industries with increasing returns to scale we should observe a small number of large firms, while with decreasing returns a large number of small firms. If both small and large companies may coexists at the same market we could imply that scale does not matter and hence industry exhibits constant returns.

The following examples discuss the introduced concepts. See also Douglas (1948) for a seminal development on production functions.

Example 3.1 (Leontief production function) *Consider a production function f with $q = f(k,l) = \min\{\alpha k, \beta l\}$, where α, β are positive. This is an example of a perfectly complementary inputs as the production is always limited by the minimum of both (perhaps multiplied) inputs. The isoquant curve is L-shaped and not differentiable at the point where $\alpha k = \beta l$. The function exhibits constant returns to scale. The elasticity of substitution is zero as there is no possibility for substitution.*

Example 3.2 (Linear production function) *Consider a linear production function f given by $f(k,l) = \alpha k + \beta l$, where $\alpha, \beta > 0$. This is an example of a perfectly substituable inputs and the isoquant is linear. The $MRTS_{k,l} = \frac{\alpha}{\beta}$. The function exhibits constant returns to scale and the elasticity of substitution is ∞.*

Example 3.3 (Cobb-Douglas production function) *Let* $f(k,l) = k^{\alpha}l^{\beta}$*, with parameters* $\alpha, \beta > 0$*. Then* $MP_k = \alpha k^{\alpha-1}l^{\beta}$ *and* $MP_l = \beta k^{\alpha}l^{\beta-1}$*. Parameter* α *measures the elasticity of production with respect to* k*, i.e.* $\varepsilon_k^f = \frac{kMP_k}{f} = \frac{\alpha k k^{\alpha-1}l^{\beta}}{k^{\alpha}l^{\beta}} = \alpha$*. The* $MRTS_{k,l} = \frac{\alpha l}{\beta k}$ *and the elasticity of substitution is* 1*. The function exhibits the constant returns to scale if* $\alpha + \beta = 1$*, while increasing (decreasing), if* $\alpha + \beta > (<) 1$*.*

Example 3.4 (CES production function) *Let* $f(k,l) = [\alpha k^{\frac{\sigma-1}{\sigma}} + \beta l^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$ *for* α, β, σ *positive. Observe that for* $\sigma = 0$ *the production function becomes Leontief, if* $\sigma = 1$ *then production function becomes Cobb-Douglas, while for* $\sigma = \infty$ *production function becomes linear. The* $MRTS_{k,l} = \frac{\alpha k^{\frac{-1}{\sigma}}}{\beta l^{\frac{-1}{\sigma}}}$*. Rearranging:* $\frac{k}{l} = (\frac{\alpha MRTS_{l,k}}{\beta})^{\sigma}$ *and* $\ln\frac{k}{l} = \sigma(\ln\frac{\alpha}{\beta} + \ln|MRTS_{l,k}|)$*. Hence the elasticity of substitution* $ES = \frac{\ln\frac{k}{l}}{\ln|MRTS_{l,k}|} = \sigma$*.*

Observe that in the examples above marginal productivity of labor (capital) is increasing (or nondecreasing) with the amount of capital (labor) used. This means that capital and labor are complementary in the production process, but this feature could be generalized to many inputs example. **Complementarities** (or "strategic fit") are now considered to be a very important factor in shaping the production process and understanding modern manufacturing (see Kremer (1993) and Milgrom and Roberts (1990,1995)).

Finally, we speak about a **technological progress**, if a change in the technology / production function allows to produce more from a given number of inputs. In such a case the isoquant is shifted towards the origin of the graph with inputs as variables. If the shift leaves MRTS unchanged we speak of a **neutral technological progress**. But if the $MRTS_{k,l}$ decrease after a technological change, we say that technological progress is **capital saving**. Intuitively, if $MRTS_{k,l} = \frac{MP_k}{MP_l}$ decrease it means that MP_l increases faster than MP_k, hence labor becomes relatively more productive and proportionally some amount of capital could be saved. Similarly, if $MRTS_{k,l}$ increases after a technological change, we say that technological progress is **labor saving**.

3.2 Costs

Before we analyze the production cost minimization problem we introduce some basic costs concepts. The **economic costs** are equal to **accounting costs** plus the **alternative costs**. The alternative costs are implicit and account for the opportunity cost (of working time or alternative capital use etc.), that is the value of the next best alternative that could be done, if the current activity is surpassed.

Economists also differentiate between **sunk** and **nonsunk costs**. Sunk costs are nonavoidable while nonsunk are avoidable. Usually the sunk costs are irrelevant for economic analysis as they are nonavoidable and hence will not influence one's decision.

Consider now a problem of minimizing the costs of producing output q given technology f. Assume that price of a unit of labor is w (e.g. hourly wage of unskilled worker) while renting price of capital is r. The total (minimum) cost of producing q is hence:

$$TC(q) = \min_{k,l \geq 0} wl + rk,$$
$$\text{s.t. } f(k,l) \geq q.$$

The solutions to this minimization problem are called input demand functions and are denoted by $l^*(q,w,r), k^*(q,w,r)$. For quasiconcave and differentiable f, optimal (and interior) solution to this problem is characterized by:

$$\frac{MP_k}{r} = \frac{MP_l}{w},$$

or equivalently: $MRTS_{k,l} = \frac{r}{w}$, hence the rate at which the firm wants to exchange inputs along the isoquant must be equal to the rate at which the market can exchange both inputs[3]. For corner solutions the condition may not hold. However, since there are only two inputs considered in the optimization problem, there are only two corner solutions, which need to be verified, i.e. $(k = 0, l > 0)$ or $(k > 0, l = 0)$. Hence, one need to compare the costs of wl with rk such that $q = f(0,l) = f(k,0)$.

If the ratio of prices changes, typically firms' change their input employment. We can analyze this by observing how the optimal $l^*(q,w,r), k^*(q,w,r)$ vary with prices. Consider the following example.

Example 3.5 (Cost function for a Cobb-Douglas technology) *Let prices w, r be given and consider $f(k,l) = k^{.5}l^{.5}$. Solving $q = k^{.5}l^{.5}$ for k gives isoquant $k = \frac{q^2}{l}$. Putting that to the optimization problem $TC(q) = \min_{l \geq 0} wl + r\frac{q^2}{l}$, where the first order condition for interior optimal l^* is $w = r\frac{q^2}{(l^*)^2}$ and hence $l^*(q,w,r) = q\sqrt{\frac{r}{w}}$, that is decreasing in w, increasing in r, and linear in q. The total costs function is hence $TC(q) = 2q\sqrt{rw}$.*

Having derived the cost function we define **average costs** as $AC = \frac{TC(q)}{q}$ and **marginal costs** $MC = \frac{\triangle TC(q)}{\triangle q} = \frac{TC(q+\delta) - TC(q)}{(q+\delta) - q} = \frac{TC(q+\delta) - TC(q)}{\delta}$, for small increase δ in production, or simply $MC(q) = TC'(q)$ for differentiable total costs function. If some part of the total costs does not change with q (i.e. $FC = TC(0)$), we call this a **fixed costs** and write $TC(q) = FC + VC(q)$ or $AC(q) = \frac{FC}{q} + \frac{VC(q)}{q}$. Notice that fixed costs can be either sunk or not. Can you think of some examples?

[3] Recall the condition for optimal consumption bundle in the consumer choice problem, discussed in chapter 2.2. Again, the similarity of the two conditions will be important in the general equilibrium analysis.

Consider the total costs function $TC(q; w, r) = wl^*(q, w, r) + rk^*(q, w, r)$. What is the change of the total costs if factor prices (e.g. w) change? Observing the above formula suggest, that we shall account for the direct effect of a price change, as well as for an indirect effect via $w \to l^*(q, w, r)$ and $w \to r^*(q, w, r)$. **Shephard's lemma** assures that for differentiable factor demand functions the indirect effects cancel out, and:

$$\frac{\partial TC}{\partial w}(q; w, r) = l^*(q, w, r).$$

This follows from envelope theorem and imply that a rate of change of the total cost function with respect to input price is equal to the corresponding input demand function. See Shephard (1978) for a formal treatment.

Interestingly[4] the AC is increasing if $MC > AC$ and decreasing if $AC > MC$. Similarly $MC = AC$ at the minimal AC. We call level of q for which AC is minimized a **minimal efficient scale**. If AC is decreasing (increasing) we speak of **economies (diseconomies) of scale**. Clearly increasing returns of scale are equivalent to economies of scale, while decreasing returns are equivalent to diseconomies of scale. See Stigler (1951) for a classic reference.

If all inputs are variable, we say that $TC(q)$ is the **long run total cost** and denote it by TC_{LR}, while if some factor is fixed (usually capital $\bar{k}$), then the $TC(q) = wl + \bar{k}r$ is the **short run cost curve**, with $q = f(\bar{k}, l)$ and notation TC_{SR}. Clearly $TC_{SR} \geq TC_{LR}$ as the long-run cost function is an envelope of the short-run one. Similarly we can define short- or long-run average/marginal costs.

Two important concepts related to economies of scale are economies of scope and economies of experience (also dynamic economies of scale or learning-by-doing effect). If we consider production process producing two outputs q_1, q_2 and derive a cost function $TC(q_1, q_2)$, we speak of **economies of scope**, if $TC(q_1, q_2) \leq TC(q_1, 0) + TC(0, q_2)$, that is, it is cheaper to produce both outputs at the same time than separately[5]. It implies complementarities in production process and a typical example of such a pair is energy and pollution.

Economies of experience capture dynamic learning effect. Specifically, if average cost $AC(Q_t, q_t)$ is nonincreasing with $Q_t = \sum_{k<t} q_k$, which measures the total output produced till period t, we say that technology exhibits economies of experience, as it is cheaper to produce the next unit, then the previous one. Observe that $q_t \to AC(Q_t, q_t)$ may be increasing or decreasing, hence economies of experience are not related to economies of scale (see Arrow, 1962).

We refer the reader to the textbook by Fuss and McFadden (1980) with extensions and application of the basic models covered in this chapter.

[4]Trivially $AC(q) = \frac{TC(q)}{q}$ hence $AC'(q) = \frac{qMC(q) - TC(q)}{q^2}$ and the desired inequalities follow if we set $AC'(q) \lesseqgtr 0$.

[5]See also Topkis (1995), who analyze comparative statics of a firm, for some examples.

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Chapter 4

Perfect competition

4.1 Company at the perfectly competitive market

Here we start with the analysis of a firm's behavior at the perfectly competitive market. Firm is taking a market price $\overline{p}$ as given and beyond its control (see Robinson (1934) and Stigler (1957) for a seminal discussion of the basic assumptions underlying perfect competition and its methodological development). The demand it faces is hence given by:

$$D(p) = \begin{cases} 0 & \text{if } p > \overline{p}, \\ \text{any amount} & \text{if } p = \overline{p}, \\ \infty & \text{if } p < \overline{p}. \end{cases}$$

The firm can set any price it wants but if the price is higher that the market one $\bar{p}$ it will sell nothing, while if the price is lower than $\bar{p}$ firm has an infinite demand. Still the profits would be higher, if the price is actually equal to $\bar{p}$, as then a firm may still have as many clients as it wants. The profit maximization problem is hence simple and requires to find the appropriate level of production to maximize revenues minus total costs at the market price, say p:

$$\max_{q \geq 0} pq - TC(q),$$

where function TC was derived in the previous chapter. Assuming TC is differentiable and the second order condition $TC''(q^*) \geq 0$ is satisfied, we have the necessary and sufficient condition for interior, profit maximization quantity level q^* is:

$$p = TC'(q^*) = MC(q^*).$$

Observe that in some cases, like with fixed costs, the interior solution may not be optimal. To see, when a corner solution is chosen, consider $TC(q) = VC(q) + FC$

with $FC > 0$ that are sunk. Then condition 'price equal to marginal costs' is still necessary, but not sufficient unless $\pi^* = pq^* - VC(q^*) - FC \geq -FC$. This says that profit must exceed profit of producing zero ($-FC$). And that yields condition: $p \geq \frac{VC(q^*)}{q^*} = AVC(q^*)$. The price level at which $p = AVC_{min}$ is called a **shutdown level** as the company would be better off closed. For the case, when the fixed costs are nonsunk the condition becomes $\pi^* = pq^* - VC(q^*) - FC \geq 0$ or $p \geq ATC$. That is, as the nonsunk costs are avoidable, then firm would continue production until profits are positive.

Some costs are sunk in the short run, but in the long run most costs become nonsunk. Hence $p \geq AVC(q^*)$ is a condition for firm's positive operations in the short run, while $p \geq ATC(q^*)$ in the long run. The intermediate cases, where some fixed costs are sunk, while others not, can also be considered.

Observe that condition $p = MC(q^*)$ implicitly defines a supply function $S(p)$ by $p = MC(S(p))$ with $TC''(S(p)) \geq 0$ as long as (short or log run) operating conditions are satisfied. Hence the inverse of marginal costs defines a (short or long run) **firm's supply curve**.

In the analysis so far we have assumed that TC function was derived from company costs minimization problem. Still, a direct analysis is possible. We study it now. Consider a profit maximization problem of a firm with production function f taking prices p of output and inputs w, r as given:

$$\max_{k,l \geq 0} pf(k,l) - rk - wl.$$

If the production function is strictly concave and differentiable then necessary and sufficient conditions for interior profit maximization inputs (k^*, l^*) are $f_1'(k^*, l^*) = MP_k = \frac{r}{p}$ and $f_2'(k^*, l^*) = MP_l = \frac{w}{p}$. It says that the marginal productivity of each factor is equal to its real price. Also, observe, that this condition determines both: cost minimizing inputs k^*, l^* (as it imply $\frac{MP_k}{r} = \frac{MP_l}{w}$), as well as the optimal production level $q^* = f(k^*, l^*)$ given price p.

Remark 4.1 *We have shown that, at the perfectly competitive market, price equals marginal cost. One may argue that it is because of a price taking assumption, which is justified with a large number of competitors. In chapter 8, however, we discuss an example where this result still holds for $m = 2$ firms that are price setters.*

4.2 Competitive equilibrium and welfare

In the previous section we have derived a supply curve S of a firm at the perfectly competitive market. If we sum supplies of all m firms we obtain a **market** or **aggregate supply**: $\sum_{j=1}^{m} S_j(p)$. Similarly in the previous chapter we obtained an aggregate (or market) demand $\sum_{i=1}^{n} D_i(p)$ of some n consumers. A pair (q^*, p^*) is called a **competitive (partial) equilibrium** if:

$$q^* \in \sum_{j=1}^{m} S_j(p^*) \text{ and } q^* \in \sum_{i=1}^{n} D_i(p^*).$$

In such a case there is a price p^* for which quantitiy q^* is supplied and demanded. If demand and supply are functions this condition simply says:

$$\sum_{j=1}^{m} S_j(p^*) = q^* = \sum_{i=1}^{n} D_i(p^*),$$

which equates supply to demand. This is the condition that determines a market price. Observe that in equilibrium all m firms may have positive or zero profits.

For a given demand and supply a natural question is, whether there exists a price such that market clears. We postpone discussing answers to this questions till chapter 9. Here we only mention that indeed the market equilibrium may or may not exists. Moreover, if it exists its uniqueness is not guaranteed, even if demand and supply are functions (see figure 1.1).

The long run perfectly competitive analysis endogenize m. That is: in the **long run perfectly competitive equilibrium** m^* is such that

$$\sum_{j=1}^{m^*} S_j(p^*) = \sum_{i=1}^{n} D_i(p^*),$$

and

$$(\forall j) \quad AC(S_j(p^*)) = MC(S_j(p^*)) = p^*.$$

Interpreting: in the long run competitive equilibrium the profits of all companies are zero. This must hold since, as if they were positive, some new companies could appear, enter the market and reduce them.

We now move to welfare analysis of competitive equilibrium. For this reason assume there is a representative consumer with quasilinear, differentiable utility function $u(x) + y$, where y is simply money left for purchase of other goods, and it is the market for good x that is under study. We already know that demand for x is given implicitly by $u'(D(p)) = p$, where p is a price of x. Assume also that a total cost function of producing x is given by TC with $TC''(\cdot) > 0$ and $TC(0) = 0$. Then supply of x is given implicitly by $TC'(S(p)) = p$. In the competitive equilibrium we hence have:

$$u'(D(p^*)) = p^* = MC(S(p^*)).$$

Now consider a **welfare maximization problem** of:

$$\max_{x,y \geq 0} u(x) + y,$$
$$\text{s.t. } y = e_y - TC(x),$$

where e_y is an initial endowment in good y. This problem maximizes utility of a representative consumer under the feasibility constraint, and yields a first order condition for interior solution:

$$u'(x^*) = MC(x^*),$$

that is also sufficient under our assumptions. Hence the competitive equilibrium allocation maximizes total welfare as the price p^* equates values of the two functions.

To see it from a different perspective, observe $TC(x) - TC(0) = \int_0^x MC(q)dq$. Similarly observe that $u(x)$ is simply an area under the inverse demand function. Recall that consumer surplus is $CS = u(x) - px$ and producer's surplus is $PS = px - TC(x)$. As a result the competitive equilibrium allocation also maximizes a **total surplus**: $PS + CS$, i.e.

$$\max_{x \geq 0} PS + CS = \max_{x \geq 0} u(x) - px + px - TC(x).$$

These properties of a competitive equilibrium may not hold, however, when prices are influenced by some taxes / subsidies or quotas. Finally we will discuss competitive equilibrium in a general equilibrium setting in chapter 9.

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Chapter 5

Monopoly and pricing

A monopoly is a firm with market power, i.e. it observes impact of own quantity change on the market price. Hence we say that monopoly is a price maker. Observe that in order to determine, whether a company is a monopolist or not, one needs an appropriate definition / boundaries of a market. Monopolists face two constraints: technological summarized by a cost function TC, and market summarized by demand D. The problem of monopolists is to choose price and quantity to maximize profits under these two constraints:

$$\begin{aligned} &\max_{p,q \geq 0} pq - TC(q), \\ &\text{s.t. } q \leq D(p). \end{aligned}$$

In many cases it requires that $D(p) = q$ (when it is not?) and hence the problem reduces to:

$$\max_{p} pD(p) - TC(D(p)).$$

Also, if D is invertable (with inverse demand function P) we can alternatively express the monopolist problem as:

$$\max_{q \geq 0} P(q)q - TC(q).$$

For differentiable P and TC functions, the first order conditions for interior solution imply:

$$P'(q^*)q^* + P(q^*) = TC'(q^*),$$

which reads that **marginal revenue** is equal to marginal costs. Inspecting this equation we see than the monopolist increasing output balances cost change effect (MC) with increased revenue of selling more at the current price (P) and (typically) decrease in revenue, as to sell more he needs to decrease the price of every unit offered ($P'(q)q$). The second order condition is:

$$2P'(q^*) + q^*P''(q^*) - TC''(q^*) \leq 0.$$

This first order condition can be rearranged to give:

$$P(q)[1 + \frac{P'(q)}{P(q)}q] = TC'(q),$$

or

$$\frac{P(q) - TC'(q)}{P(q)} = -\frac{P'(q)}{P(q)}q = -\frac{1}{\epsilon_p^D}.$$

The right hand side term is a (minus) inverse of price elasticity of demand and gives the simple rule for monopoly pricing. The left hand side is sometimes referred to as the **Lerner index**, measuring the % of price that finances markup (difference between price and marginal costs). Also, it follows that at the optimal level of quantity the price elasticity of demand must be greater than one in absolute values.

To challenge your thinking try to analyze the following case. Suppose marginal costs of production increase as a result of technological change, or market regulations. When is it a case that the monopoly will increase its absolute markup? How about its relative markup?

We now present few examples or deriving the optimal monopoly price:

Example 5.1 (Linear demand and costs) *Let $TC(q) = cq$ and assume linear demand with inverse $P(q) = a - bq$. Then the optimal production satisfies $-bq^* + a - bq^* = c$ or $q^* = \frac{a-c}{2b}$ with $P(q^*) = \frac{a+c}{2}$.*

Example 5.2 (Constant elasticity of demand) *Let $TC(q) = cq$ and assume constant price elasticity of demand function $D(p) = Ap^{-b}$. In such case the monopoly pricing rule gives: $p^* = \frac{c}{1-\frac{1}{b}}$.*

Example 5.3 (Monopsony) *The analysis of the monopolistic behavior is not restricted to the case of a company selling a good to customers. Similarly one can analyze behavior of a company that is a single buyer of e.g. labor services at some market. To see that formally let f be a production function from (a single input) labor and w be an inverse labor supply curve. A single firm buying labor services on that market that sees its impact on the wage offered is called a* ***monopsony****. Formally a monopsony problem is to choose a labor input such that*

$$\max_{l \geq 0} pf(l) - w(l)l,$$

where we assume that the firm is price taker on the consumption good market. Similarly as before, assuming differentiability of the objective, the first order condition for *interior choice of* l^* *is* $pf'(l^*) = w(l^*) + w'(l^*)l^*$ *or rewriting* $\frac{w(l^*)-pMP_L}{w(l^*)} = -\frac{1}{\epsilon_w^L}$, *where* ϵ_w^L *is a wage elasticity of labor supply. Observe that mixed cases are also possible allowing for company that has market power at both the input and output markets for example.*

Typically $P'(q) < 0$ and together with increasing marginal costs, a monopoly pricing rule implies that monopolist would produce less than a perfectly competitive firm. It also implies higher prices set by monopolist. This indicates that monopolistic solution is not socially efficient. To see that clearly, consider a single consumer economy with quasilinear utility $u(x)+y$ giving an inverse demand function $p(x) = u'(x)$. There is also a monopolist with differentiable cost function TC. The social objective is to maximize $W(x) := u(x)-TC(x)$ which gives the first order condition $u'(x^*) = p(x^*) = MC(x^*)$. On the other hand the monopolist chooses $p(x_m) + p'(x_m)x_m = MC(x_m)$ and hence $W'(x_m) = u'(x_m) - MC(x_m) = -p'(x_m)x_m = -u''(x_m)x_m > 0$ if u'' is negative. The inefficiency of a monopoly is sometimes called a **deadweight loss**.

The next question we consider is: why (generally inefficient) organizations as monopolists are present in the market. The starting point concerns the so called **natural monopolies**. That is the case, where a single firm is more efficient (has lower average costs for all production levels at which inverse demand is higher than average costs), than two or more firms operating separately. The typical example is a company with decreasing average costs function that is characterized by economies of scale for all relevant output levels (i.e. output levels such that inverse demand is higher than average costs).

More generally there are **barriers to entry** that restrict new companies to enter the market and to challenge the monopolist. Following a classification introduced by Bain we consider structural and strategic barriers to entry. **Structural** barriers to the entry exist, when incumbent firms have cost or demand advantages that would make it unattractive for a new firm to enter the industry. One reason for that could be some form of technological effects, like natural monopoly, but it may also include some legal regulations making it very costly to enter. Alternatively **strategic** barriers to entry include incumbent firm taking explicit steps to deter the entry. The examples include limit pricing (decreasing own price to signal low markups and deter entry), increasing production to move down the experience curve (learning-by-doing effect), deliberately increasing customers' switching costs or developing networking effects via lowering costs of within network consumption. Although inefficient a monopoly makes a higher profit, than a perfectly competitive firm, hence monopolistic firms often engage in **rent-seeking** activities to protect their market power and increase profit (fortunately not always at the costs of efficiency) (see Stigler (1971) for an early introduction to the economics of regulation or Pepall, Richards, and Norman (2008) for some more recent developments).

We have stated before that monopolistic production choice is socially inefficient.

One can think that this results from a market power assumption. However, even with a market power the efficient solution is possible. This can be achieved or approximated by various **price discrimination** techniques. This again suggests that inefficiency may result also from an implicit assumption that monopolist claims only a single (and linear) price.

The **first degree** (or perfect) price discrimination means that a seller charges different prices for every unit of the good sold, such that the price for every unit equals the maximal willingness to pay. The **second degree** price discrimination (or nonlinear pricing) means that the seller asks a different price for each unit sold but do not differentiate prices between customers. The examples include quantity discounts. The **third degree** price discrimination means that different prices are offered for separate groups (segments) of clients but each segment gets a linear price. The examples include students/senior discounts but also charging different prices for business and economy class airplane tickets. The distinction between all three price discriminations is neither exhaustive nor mutually exclusive and moreover real life examples are using many of them simultaneously.

Now in a series of examples (borrowed from Pepall, Richards, and Norman (2008)) we illustrate the role of various types of pricing strategies. In examples 5.4-5.6 we consider an economy with two groups (segments) of clients with quasi-linear utilities (young and old, denoted by y, o respectively), with inverse demand for some product in each group given by $p_o = 16 - q_o$, $p_y = 12 - q_y$. The marginal costs are constant and equal to average costs $c = 4$. In the example 5.4 we consider a benchmarking case of perfect competition and (linear) monopolistic price.

Example 5.4 (Perfectly competitive and monopolistic prices) *The total demand for a product is given by* $q := q_o + q_y = 16 - p + 12 - p$ *giving the inverse total demand function:* $p = 14 - \frac{q}{2}$. *A perfectly competitive price is* $p^{PC} = c = 4$ *and perfectly competitive quantity* $q^{PC} = 20$ *with zero profit.*

Conversly, at the monopolistic market the optimality condition requires equating marginal revenue with marginal costs giving: $14 - \frac{q}{2} - \frac{q}{2} = MR = c = 4$ *resulting in* $p^M = 9, q^M = 10$ *with profits* $\pi^M = 50$. *For further reference observe that* $10 = q^M = q_o^M + q_y^M = 7 + 3$.

In the next example we allow the monopolist to discriminate its price between two groups of consumers and hence consider a case of the 3rd degree price discrimination.

Example 5.5 (Observable characteristics) *Consider a monopolist choosing two separate prices for each group of clients. We hence analyze the optimality conditions equating costs and marginal revenue at each market separately. That is* $4 = c = MR_o = 16 - q_o - q_o$ *and* $4 = c = MR_y = 12 - q_y - q_y$ *giving* $q_o = 6$ *and* $q_y = 4$ *with prices* $p_o = 10$ *and* $p_y = 8$. *The total profit of the monopolist is hence equal* *to* $36 + 16 = 52$. *The profits are higher than under the monopolistic solution in the example 5.4, although the total output produced is the same.*

As the next example illustrates the 3rd degree price discrimination proposed in the example 5.5 is not socially optimal. That is, there exist a better tariff combining the second and third degree price discrimination, simultaneously implementing the perfect price discrimination solution.

Example 5.6 (Optimal two part tariff) *Consider the following two part tariff for each group of clients. The unit price for each group is the same and equal marginal costs* $p_y = p_o = 4 = c$ *but on top of that the monopolist asks to pay a constant/fixed fee* t *(not changing with the quantity consumed) to be allowed to consume this good. Let* $t_o = 72$ *and* $t_y = 32$. *Observe that under such tariff the old consumers consume* $q_o = 12$ *units but their surplus from such consumption is zero, i.e.* t_o *captures the whole surplus from consumption of* 12 *units each at price* 4. *Similarly the young consume* $q_y = 8$ *and* t_y *is chosen so that the surplus of the young is zero. The total output produced in* $q = q_y + q_o = 20$ *and gives profit* $32 + 72 + (4 - 4) \times 20 = 104$, *which is the highest possible in this examples. Hence two-part tariff can implement the first degree price discrimination. Observe that in such a case the amount (denoted by* M*) the consumer spends to buy a given number of goods* q *is nonlinear in* q. *Specifically:*

$$M(q) = \begin{cases} 0 & \text{if} \quad q = 0, \\ qp + t & \text{if} \quad q > 0. \end{cases}$$

In the next example we again consider the case with two customer groups, but now (as opposed to age) the characteristics differentiating both groups are unobservable (e.g. think of price discrimination based on income). So let $p_h = 16 - q_h$ and $p_l = 12 - q_l$, where h, l stands for high and low income respectively.

Example 5.7 (Unobservable characteristics) *Suppose we want to introduce an optimal two part tariff from the example 5.6. Observe now that, if the company introduces two separate tariffs, but cannot easily differentiate its consumers, these are the customers that will choose a tariff that fits them best. Specifically, consider the tariffs from the example 5.6:* $p_l = 4$ *and* $t_l = 32$ *and* $p_h = 4$ *and* $t_h = 72$. *Now, low income customers will choose the* l *tariff giving them zero surplus, but the high income consumers instead of consuming the* h *tariff (giving them zero surplus) are better off choosing tariff* l, *consuming* 8 *and giving the positive surplus of* $64 - 32 = 32$. *In such a case we say that tariff* h *does not satisfy the* ***incentive compatibility*** *constraints, as the* h *clients can misreport their income, pretend they are from* l *group and be better off. The optimal tariff for* h, *constrained by incentive compatibility, yields:* $p_h = 4$ *and* $t_h = 40$. *In such a case* h *clients are indifferent between* l, h *tariffs and can choose to take* h. *This gives (constrained) maximal profit of* $40 + 32 = 72$.

Similar consideration can be taken into account, when ones tries to **discriminate prices in time**, i.e. set high prices in the beginning of sale of some good, and decrease it in time to capture the consumers that can wait longer. Of course the incentive compatibility must be taken into account (see Bulow, 1982).

There are other types of pricing strategies including: block pricing or bundling. **Block pricing** is an example of the second degree price discrimination in which the firm offers quantity discounts. An example with two blocks looks like this:

$$M(q) = \begin{cases} qp_1 & \text{if} \quad q \leq \bar{q}, \\ \bar{q}p_1 + (q - \bar{q})p_2 & \text{if} \quad q \geq \bar{q}, \end{cases}$$

with $p_1 > p_2$. **Bundling**, or more generally tying, refers to situations with multiple goods. The next example illustrates this. More on bundling can be found in (Adams and Yellen, 1976).

Example 5.8 (Bundling) *Suppose we have 4 groups of customers (each of equal size 1) each with willingness to pay for goods A and C presented in the next table. Let the cost of production be equal to 30.*

Goods	*A*	*C*
1	*20*	*100*
2	*40*	*80*
3	*80*	*40*
4	*100*	*20*

If a company chooses prices of both goods separately, then the profit maximization yields: $p_A = 80$ *and* $p_C = 80$ *giving profits* $2 \times 80 + 2 \times 80 - 4 \times 30 = 200$, *as only two groups of customers would buy each product. Now suppose that the company sells bundles of goods A and C, with a price of such bundle* $p_B = 120$. *Then each group buys both products giving profit* $4 \times 120 - 8 \times 30 = 240$. *Finally consider the* ***mixed bundling****, in which there is a price of the bundle* $p_B = 120$ *and one can also buy each product at* $p_A = p_C = 99$. *Observe that in such a case consumers from group 2 and 3 buy a bundle, while consumers from group 1 and 4 are better off buying only one product and enjoying surplus of* 1. *In such a case the company's profit is* $2 \times 99 + 2 \times 120 - 6 \times 30 = 258$. *Can you generalize intuition on, when bundling increases profit of a company, and when mixed bundling increase it even further?*

Tying is a more general form of bundling, and refers to the case, when the consumption of one product requires consumption of the other (e.g. camera and appropriate lenses that fit).

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Chapter 6

Risk and ambiguity

In this chapter we consider decisions under uncertainty. More specifically we will analyze decisions over risky or ambiguous outcomes. As this may be useful to analyze both decisions of a consumer or a firm we will generally talk on some decision makers. Gilboa (2009) textbook is a great reference for topics discussed in this chapter.

6.1 Expected utility

In this section we consider decisions under uncertainty over the finite set of outcomes X. By a **lottery** we mean[1] a probability distribution on X. If X has cardinality n we can think of $f \in X$ as of vector of nonnegative numbers $f_1, f_2, \ldots, f_n$ such that $\sum_{i=1}^{n} f_i = 1$. By L denote the set[2] of all lotteries on X. We now define a preference $\succeq$ on L that we assume is rational (i.e. complete and transitive) and continuous. Here we just mention that rationality assumption is the more restrictive the more complicated the domain of choice is. Hence rationality assumption is sometimes questioned at this level. However the critical assumption on preference $\succeq$ concerns the so called **independence**. Specifically for all $f, g, h \in L$ and all $\alpha \in [0, 1]$ we have

$$f \succeq g \text{ iff } \alpha f + (1-\alpha)h \succeq \alpha g + (1-\alpha)h.$$

To understand the independence observe that $\alpha f + (1-\alpha)h$ can be seen as a mixed lottery (with probabilities α) over f and h. Hence it is naturally to think that $\alpha f +$

[1]Alternatively, we can define a lottery as a function ℓ mapping states $s \in S$ to outcomes $x \in X$, i.e. $\ell : S \to X$. Assuming, that there exists some probability distribution $p : S \to [0, 1]$, each outcome $\ell(s)$ occurs with probability $p(s)$. Therefore, both representations are equivalent. However, representing the lottery as a function might be useful, when we are willing to compare two different lotteries, defined over the same state space. For example, lottery ℓ', such that $\forall s \in S, \ell'(s) > \ell(s)$ has always higher outcomes, regardless of the state.

[2]Therefore, consider X as a set of all vectors, which coordinates sum up to 1.

$(1-\alpha)h \succeq \alpha g+(1-\alpha)h$ whenever $f \succeq g$, as a result of **consequentialism**. That is, it is either f or h that will happen/be consumed (never both), so the compound lottery h should be irrelevant for preference between f and g. Note, that this in particular means, that $f \succeq g$ if and only if $\alpha f + (1-\alpha)g \succeq g$, and $f \succeq \alpha f + (1-\alpha)g$. Therefore, the axiom implies that the preferences relation is preserved for any linear combination of the lotteries in question. Having that we can state the expected utility theorem.

Theorem 6.1 (von Neumann-Morgenstern) *Suppose $\succeq$ is rational, continuous preference relation satisfying independence. Then there exists numbers u_i (unique up to affine transformation) for each element of X such that*

$$(f_1, f_2, \ldots, f_n) = f \succeq g = (g_1, g_2, \ldots, g_n) \textit{ iff } \sum_{i=1}^{n} f_i u_i \geq \sum_{i=1}^{n} g_i u_i.$$

The von Neumann and Morgenstern (1944) theorem shows that the utility function we can use to evaluate (represent) lotteries take a very simple form of a linear function, weighting utilities of outcomes u_i with probabilities of these outcomes f_i. More generally, if we consider some random variable taking values $x \in X$ with probability $p(x)$, then the **expected utility** of choosing such a random variable is simply $\sum_i u(x_i)p_i$ or $\int_X u(x)p(x)dx$, where we set $u(x_i) := u_i$. Observe that although the expected utility is linear in probabilities it is not necessarily linear in outcomes, i.e. $x \to u(x)$ need not be linear. That is to say that expected utility is not necessarily an expected value of a random variable taking values in X. This observation is critical to measure risk. Moreover although the utility u is defied on the set of outcomes X, that in principle could be very general, and correspond to a consumption set, we often focus on X as representing the wealth levels and u as an indirect utility function. As we will mention later one needs to be cautious, though, when utilities are not defined over monetary payoffs.

Example 6.1 (Demand for insurance) *Consider a consumer with initial wealth w facing a risk of loosing l with probability p. His expected utility from such a lottery is $pu(w-l) + (1-p)u(w)$. There is also an insurance company selling policies to cover a loss of q at price (premium) πq. To analyze the demand for such insurance consider a maximization problem $\max_{q \geq 0} pu(w-l+q-\pi q) + (1-p)u(w-\pi q)$. The fact that price $-\pi q$ appears in both outcomes means that the premium must be paid in advance of resolution of uncertainty, however, cover q is only present in case of a loss. The first order condition for optimal cover q^* is then*

$$\frac{pu'(w-l+q^*-\pi q^*)}{(1-p)u'(w-\pi q^*)} = \frac{\pi}{1-\pi},$$

equating marginal rate of substitution between states with the relative price of an additional unit of a cover.

Now suppose also that an insurance company operates at the competitive market such that its economic profit is zero: $0 = \pi q - pq$*. This implies that for given p insurance premium* $\pi = p$*, which is sometimes called a* ***fair insurance price****.*

If indeed $\pi = p$*, then the consumer's first order condition reduces to* $u'(w - l + q^* - \pi q^*) = u'(w - \pi q^*)$*, yielding* $q^* = l$ *whenever* u' *is strictly monotone. Hence under a fair insurance price a full coverage of a loss is optimal. The results of this example must be analyzed with care, as we will show later that neither the zero profit condition on an insurance market need not be satisfied nor the assumption that the probabilities of events are independent of decision maker actions.*

6.2 Evaluating risk

Consider a decision maker that prefers a sure event of obtaining an expected value of some random variable taking values in $\mathbb{R}$ to an expected utility of choosing such random variable i.e.

$$u(\sum_i x_i p_i) > \sum_i u(x_i) p_i.$$

We call such person **risk averse**. If the reverse holds, we say he is a **risk lover**. If both expressions are equal we say she is a **risk neutral**. In the following discussion we focus on risk aversion. Comparing both sides of the expression above suggest that the appropriate risk measure should account for concavity of u. As this will be demonstrated later generally, mean and variance of a random variable are not enough to describe decision maker behavior nor his risk attitude. Indeed as showed by Arrow and Pratt the appropriate notion of **(absolute) risk aversion measure** (for a twice differentiable) utility function is

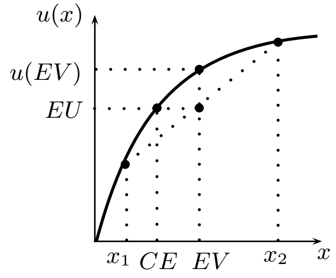
$$r(x) = -\frac{u''(x)}{u'(x)}.$$

By saying appropriate we mean that decision maker i would take more (small) gambles than j, if and only if she has a higher (Arrow-Pratt) risk aversion measure. By small we mean that the Arrow-Pratt measure is a local measure and may change with respect to a 'wealth' level x. To understand the concept of risk aversion consider a notion of **certainty equivalence** of some random variable:

$$u(CE) = \sum_i u(x_i) p_i.$$

That is, CE is a level of 'wealth' that gives the same utility as expected utility of a lottery. To see usefulness of this notion consider the decision maker faced with a lottery. The difference between the expected value of a lottery and its certainty equivalence $RP = EV - CE$ measures how much (in maximum) a decision maker is

Figure 6.1: Utility over monetary outcomes for a risk averse consumer.



willing to pay to sell risk associated with a lottery. This is called a **risk premium**. All in all, Pratt's theorem establishes that a decision maker with higher risk aversion measure has 'more concave' utility function or equivalently is willing to pay more to sell a risk of a lottery allowing him to win or loose some amount with equal probabilities. Figure 6.1 presents the example of a utility over monetary outcomes for a risk averse consumer, and a lottery to get x_1 or x_2 with some probabilities.

A related concept of measuring risk concern the so called **relative measure of risk aversion** defined by:

$$\rho(x) = -\frac{xu''(x)}{u'(x)},$$

where word *relative* refers to a multiplication by x. See also Kihlstrom and Mirman (1974) for a formal analysis of risk over multidimensional outcomes.

Example 6.2 (CRRA) *Consider a utility function* $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$*. Its absolute risk aversion is* $r(x) = \frac{\sigma}{x}$ *and relative risk aversion is* $\rho(x) = \sigma$*, hence it is called a constant relative risk aversion (CRRA) utility function and* σ *measures relative risk aversion.*

Example 6.3 (CARA) *Consider a utility function* $u(x) = -e^{-\sigma x}$*. Its absolute risk aversion is* $r(x) = \sigma$ *and relative risk aversion is* $\rho(x) = x\sigma$*, hence it is called a constant absolute risk aversion (CARA) utility function and* σ *measures absolute risk aversion.*

Example 6.4 (Mean-variance utility) *We argued that the mean and variance are not sufficient measures to capture the choice of a decision maker under uncertainty. This is true, however, there exist a special class of utility function, where these two measures are sufficient. To see that consider a utility function* $u(x) = -e^{-\sigma x}$*. Suppose that outcomes* $x \in X$ *are distributed according to density function* p*. Then the expected utility is equal* $Eu(x) = -\int_X e^{-\sigma x} p(x)dx = -e^{-\sigma(\bar{x} - \sigma\frac{\sigma_x^2}{2})}$*, where* $\bar{x}$ *is the mean and* σ_x^2 *is a variance.*

6.3 Subjective probability and state dependent utility

Von Neuman-Morgerstern approach was criticized by the assumption that probabilities are exogenously given and objective, i.e. all decision makers know some 'real' probabilities of states they face. The next example highlights some of this argument.

Example 6.5 ('Allais paradox') *You are asked to choose between two lotteries:*

A: 100% chance of receiving 1m.

B: 10% chance of receiving 5m, 89% of receiving 1m and 1% of receiving nothing.

Write down your choice. Then consider two other alternatives:

C: 11% chance of receiving 1m and 89% of receiving nothing,

D: 10% chance of receiving 5m, 90% of receiving nothing.

Again please write down your choice. Allais observed that many people prefer A over B but D over C. Such choice violates expected utility theory. To see that observe that utility function representing such choices must satisfy:

$$u(1m) \geq .1u(5m) + 0.89u(1m) + .01u(0).$$

Rearranging: $.11u(1m) \geq .1u(5m) + .01u(0)$ *and adding* $.89u(0)$ *to each side:*

$$.11u(1m) + .89u(0) \geq .1u(5m) + 0.9u(0),$$

implying that $C \succeq D$ *by an expected utility maximizer.*

One of the explanations of Allais paradox indicated that although some objective probabilities are given people have their own probability weights that are different from objective ones. For example a very unlikely state but still with positive probability may be considered by many as having (subjective) probability of zero. Moreover, in many situations true or objective probabilities are unknown. As subjective probabilities are unobservable, it is hard to measure them. The question is then, whether we can construct both utility functions and subjective probabilities from some observed choice data.

That was noticed first by Ramsey and de Finetti in the 1930/1931 that in order to construct both objects of interest from the observed choice data over (simple) lotteries the decision maker must be risk neutral. Later in their insightful theorem Anscombe and Aumann (1963) showed that it is indeed possible to construct subjective probabilities of states by revealed preference argument, if one observes decisions over a larger domain of acts (where acts, specify an objective (lottery over a set of outcomes) for every state). A 'crowning glory' of a decision theory was developed by Savage (1954), however, who showed that if one observes choices over acts (specifying outcomes for every state), by a revealed preference argument both subjective probabilities and utility function can be constructed. This is summarized in the celebrated subjective probability theorem.

Theorem 6.2 (Savage) *Consider a set of states S and finite outcomes set X. Define a space of acts $F = \{f : S \to X\}$ and consider a preference order $\succeq$ on F. Consider the following axioms:*

1. *$\succeq$ is complete, transitive, continuous*[3] *and there exists $f, g \in F$ such that $f \succ g$,*
2. *the preference between $f, g \in F$ should depend only on the values of f and g in states that they differ,*
3. *preferences are independent of states,*
4. *subjective probabilities of states do not depend on outcomes.*

Preference $\succeq$ satisfy the above axioms if and only if there exists a nonatomic probability measure μ on S and a linear function $u : X \to \mathbb{R}$ such that for every $f, g \in F$ we have

$$f \succeq g \text{ iff } \int_S u(f(s))d\mu(s) \geq \int_S u(g(s))d\mu(s).$$

This generalized the Von-Neumann Morgerstern analysis to a great extent and allowed to rationalize much broader class of choices. However, not all decisions can be again explained as the next example suggests.

Example 6.6 (Ellsberg 'paradox') *There are two urns: K,U with 100 balls each (either white or black). The K urn contains 49 white and 51 black balls. The U urn has unspecified amount of balls. Your are to drawn one ball from either K or U. There are two choice situations*

A: If you choose a black ball you get 1m and zero otherwise.

B: If you choose a white ball you get 1m and zero otherwise.

Ellsberg observed that many people in choice situation A prefer to choose from urn K, and similarly in the choice situation B. But this violates subjective probability theory. To see that, observe that to rationalize the first choice we must have:

$$.51u(1m) + .49u(0) \geq p_b u(1m) + (1 - p_b)u(0),$$

[3] Savage did not assume any topology on S but still assumed an axiom that played a similar role as continuity.

and hence $p_w = 1 - p_b \geq 0.49$ assuming $u(1m) \geq u(0) \geq 0$. This immediately gives that $(p_w - .49)u(1m) + u(0) \geq (p_w - .49)u(0) + u(0)$ implying

$$p_w u(1m) + (1 - p_w)u(0) \geq .49u(1m) + (1 - .49)u(0).$$

And hence, in the choice situation B the subjective utility maximizer should choose to drawn from U.

A typical explanation of Ellsberg 'paradox' is to say that the decision maker prefers a 'safe' choice is a sense that, the probability distribution is known to him. This, however, suggests that the Knight's distinction between the risk (where the probability distribution is known) and uncertainty/ambiguity (where the probability distribution is unknown) may have some insight for decision theory. See also Gilboa and Schmeidler (1989) and Klibanoff, Marinacci, and Mukerji (2005) for some recent developments.

Finally an interesting generalization was proposed by Karni, Schmeidler, and Vind (1983) concerning the so called **state dependent utilities**. They propose axiomatization of such preferences and representation by a subjective probability and utility function $u(f(s), s)$, that depends on both outcomes and states directly. The uniqueness of such representation may by problematic, however. State dependent utilities are useful in considering the choice under uncertainty over acts, where Savage third assumption is not satisfied, typically for non-monetary payoffs. A typical example supporting using such preferences is that e.g. an umbrella may give different utility in different states of nature. Although this example can be modeled using the Savage's model if the set of outcomes and states is rich enough, e.g. an umbrella is a different good in every state, the Savage's theory has problems in situations, where a decision maker may change its preference relations and hence decisions over outcomes in different states of nature (e.g. serious illness of a relative). Then states dependent utility assumption is more plausible. Finally, recently Karni (2011) provided derivation of preferences over 'strategies' (specifying actions and bets for every signal/information) with action dependent probabilities.

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Chapter 7

Game theory

Game theory is a branch of mathematics devoted to a study of strategic decision making. It has found many applications in economics, political or social sciences and biology, among others. In this chapter we introduce some basic concepts of games, strategies and equilibria. They are interesting on their own but we use them extensively in the next chapter 8 devoted to the analysis of oligopolies and industrial organization. We start with strategic form games, that can be interpreted, as if decisions of all players were taken simultaneously. Then in section 7.2 we consider extensive form games useful for the analysis of dynamic or sequential games. At this level we skip a branch of game theory devoted to the analysis of cooperative games, indicating however, their usefulness in the analysis of general equilibrium concept. More on introduction to game theory can be found in Dixit, Reiley, and Skeath (2009). See also Osbourne and Rubinstein (1994), Myerson (1991) or Fudenberg and Tirole (2002) for a more formal treatment.

7.1 Strategic form games

Formally a strategic form game Γ is a triple $(N, (A_i, u_i)_{i\in N})$ such that N is a set of **players** (with a slight abuse of notation we will denote cardinality of N by N as well), A_i is a set of **actions**[1] available to player i and $u_i : \times_{i\in N} A_i \to \mathbb{R}$ is a **payoff** of player i, when players choose action profile $(a_1, a_2, \dots, a_N) \in \times_{i\in N} A_i$. We sometimes write (a_i, a_{-i}) to denote an action profile such that player i uses $a_i \in A_i$ and other (other then i) players choose $a_{-i} \in \times_{j\neq i} A_j$. Generally player's payoff may depend on the actions taken by all players. In the strategic form game actions are taken

[1] In such simple, strategic form games we use actions / strategies terms equivalently. In extensive form games or Bayesian games, when some information is revealed during the course of the game, actions and strategies denote different concepts.

'simultaneously', i.e. when choosing its own action player does not know the other players' actions. To describe a game we adopt a matrix notation, as in the following two players example.

Table 7.1: A dominant strategy game.

	L	R
U	1,2	2,1
D	3,1	5,0

This matrix describes a game between two players: 1 (row) and 2 (column), hence $N = \{1, 2\}$. Actions available to player 1 are $A_1 = \{U, D\}$ and to player 2 are $A_2 = \{L, R\}$. When player 1 chooses D and player 2 chooses R the payoff of 1 is 5 and payoff of 2 is 0, as summarized by the entry (5,0) in the row D and column R. Formally $u_1(U, L) = 1, u_2(U, L) = 2,\ u_1(U, R) = 2, u_2(U, R) = 1,\ u_1(D, L) = 3, u_2(D, L) = 1$ and $u_1(D, R) = 5, u_2(D, R) = 0$.

We now proceed to the analysis of players' behavior in the game. It is assumed that each player knows the game he/she plays (i.e. the matrix) and aims to maximize its own payoff. The unknown is the other player action as the game is a simultaneous move one.

Observe that the game depicted in table 7.1 is relatively simple to analyze, i.e. observe that player 1 has a **strictly dominant strategy**, i.e. whatever player two chooses he is (strictly) better off by choosing D, than by choosing any other strategy. Similarly observe that player 2 has also a strictly dominant strategy L. Hence we can conclude that in this game players choose profiles (D,L) giving them payoffs (3,1).

Table 7.2: Battle of sexes (or Bach-Stravinsky game).

	B	S
B	3,2	0,0
S	0,0	2,3

In fact, dominant strategies are not typical in applied games. The next example 7.2 illustrates this. Indeed in game 7.2 it is better for player 1 to choose B, when 2 plays B, while it is better to choose S, when 2 plays S. For this reason we need some other 'solution concept' to analyze such games.

Definition 7.1 (Nash equilibrium) *A pure strategy Nash equilibrium of the game* Γ *is an action profile* $(a_1^*, a_2^*, \ldots, a_N^*) \in \times_{i \in N} A_i$ *such that* $(\forall i \in N)$*:*

$$(\forall a_i \in A_i) \quad u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*).$$

Interpreting a profile a^* is a pure strategy Nash equilibrium, if no player can strictly increase its payoff by an unilateral deviation from a^*.

In game 7.2 husband and wife wants to buy tickets for Bach or Stravinsky music concert. Observe that such a game has two pure strategy Nash equilibria (B,B) and (S,S). In any case unilateral deviations are not profitable as they result in zero payoff. Hence, our first lesson is that a pure strategy Nash equilibrium does not need to be unique. The second lesson says that, in some cases it may be non-existent (see game in table 7.3).

Table 7.3: Matching pennies.

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

The game reflects a situation in which each player puts a penny on a table (choosing heads or tails) so that the other does not see it. Then they simultaneously reveal their actions. Indeed for payoffs in table 7.3 there is no pure strategy Nash equilibrium.

Let us now consider a coordination game depicted in table 7.4. There are two pure strategy Nash equilibria (1,2) and (11,22) with respective payoffs (2,2) and (4,4). Our third lesson says hence, that some Nash equilibrium payoffs may be preferred by all players to the others. The name coordination reflects the fact that players must coordinate to the one of two equilibria. Also observe that, when player 2 chooses 2, the net benefit from the action increase by player 1 (that is action change from 1 to 11) gives $0 - 2 = -2$. The same move, when player 2 chooses 22, gives $4 - 1 = 3$. Clearly $3 > -2$ and hence in this game the higher the strategy of the opponent the higher incentive to increase own strategy. In such a case, following Bulow, Geanakoplos, Klemperer, we say that the game exhibits **strategic complementarities**. If the reverse holds we say a game exhibits strategic substitutes[2].

Table 7.4: Coordination game.

	2	22
1	2,2	1,0
11	0,1	4,4

[2]It happens that games of strategic complementarities (or supermodular games) are quite common in economics (see Topkis, 1998). For a class of games with strategic substitutes we refer the reader to a paper by Dubey, Haimanko, and Zapechelnyuk (2006).

Finally we consider a celebrated prisoners' dilemma game depicted in figure 7.5.

Table 7.5: Prisoners' dilemma.

	C	D
C	-5,-5	-1,-10
D	-10,-1	-2,-2

Observe that in Prisoners' dilemma game we have a unique pure strategy Nash equilibrium (C,C) with equilibrium payoffs (-5,-5). Our forth lesson on Nash equilibrium indicates that equilibrium payoff does not necessarily give players 'optimal' payoffs. Specifically we mean that payoffs are **not Pareto optimal**[3], as there exists a different action profile (D,D) such that both players are better off. Moreover, observe that this Nash equilibrium is in strictly dominant strategies. Specifically, observer that (D,D), although giving Pareto-optimal payoff profile, is not a Nash equilibrium as every player has an incentive to deviate. This is in contrast to the coordination game in figure 7.4.

The name prisoners' dilemma corresponds to the following situation. Two men are arrested, but the police do not possess enough information for a conviction. Men are separated and each one of them is offered a proposal: if one testifies against the other (confess), and the other remains silent (does not confess), the confessor gets 1 month sentence and the other receives the 10 months sentence. If both remain silent, both are sentenced to 2 months in jail for a minor charge. If both confess, each receives a 5 month sentence. Each prisoner must choose to confess or not and decisions must be taken simultaneously.

Finally we introduce a useful concept of a best response. Formally we denote a **best response** of player i by $BR_i : \times_{j\neq i} A_j \to 2^{A_i}$, using the following maximization problem:

$$BR_i(a_{-i}) = \arg\max_{a_i \in A_i} u_i(a_i, a_{-i}). \tag{7.1}$$

That is, the best response returns a set of all actions that are maximizing payer i payoff, when others use a_{-i}. Observe that BR_i is not necessarily a function, as for some a_{-i} there could be multiple maximizers. Also it may happen that for some a_{-i} there is not best response at all. We will see such cases, when analyzing Bertrand competition game in chapter 8. Importantly, there is relation between Nash equilibrium and the joint best response correspondence BR, defined by

$$BR(a) = (BR_1(a_{-1}), BR_2(a_{-2}), \ldots, BR_N(a_{-N})),$$

[3]See definition 9.2 in chapter 9 for a formal statement.

mapping any joint strategy profile, to a vector of best responses of all players. Indeed a strategy profile a^* is a Nash equilibrium if and only if $a^* \in BR(a^*)$, i.e. is a fixed point of BR correspondence. We will use this observation in the applied games in the following chapters.

We finish with some important extension. In the proceedings we allowed the players to choose pure strategies, i.e. elements from A_i. Assume that A_i has a finite number of elements k_i. Suppose now we allow the players to choose mixed strategies, i.e. probability distribution over elements of A_i. By $\triangle(A_i)$ denote the set of all mixed strategies, i.e. $\triangle(A_i) = \{\sigma_i : A_i \to [0,1] : \sum_{j=1}^{k_i} \sigma_i(a_j) = 1\}$. We denote a typical mixed strategy of player i by σ_i and denote an expected payoff of player i under the strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_N)$ as:

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{a \in \times_{s=1}^N A_s} u(a_1, a_2, \ldots, a_N) \prod_{j=1}^N \sigma_j(a_j).$$

Definition 7.2 (Mixed strategy equilibrium) *A mixed strategy Nash equilibrium of the game Γ is an action profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_N^*) \in \times_{i \in N} \triangle(A_i)$ such that $(\forall i \in N)$:*

$$(\forall \sigma_i \in \triangle(A_i)) \quad U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*).$$

Observe that a pure strategy Nash equlibrium is a mixed strategy Nash equilibrium for degenerate lotteries.

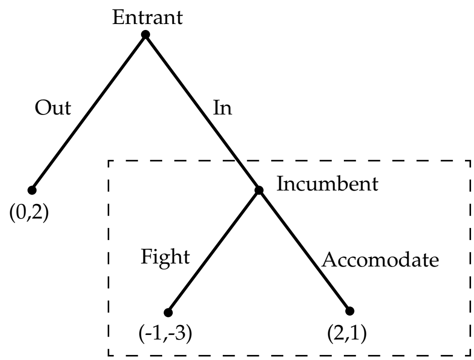
Observe that the matching pennies (see table 7.3) have a unique mixed strategy Nash equilibrium (σ_1, σ_2) with $\sigma_i(H) = \frac{1}{2} = \sigma_i(T)$ for both players. Also the battle of sexes has a mixed strategy Nash equilibrium (σ_1, σ_2), that is not pure. This is a case, when $\sigma_1(S) = \frac{2}{5} = \sigma_2(B)$ and $\sigma_1(B) = \frac{3}{5} = \sigma_2(S)$.

7.2 Extensive form games

In this section we consider extensive form games. At this level we do not introduce a formal definition of a game and strategy, but start by considering the following example depicted in figure 7.1. In this game there are two players $N = \{E, I\}$, firm contemplating entry to a particular market, and an incumbent firm. The game has two stages. In the first the entrant chooses to enter (In) or not (Out). If the firm stays out the game finishes with payoffs (u_E, u_I) =(0,2). If the firm enters, then at the second stage, the incumbent decides to accommodate entry (A) or fight (F) with an entrant. In each case the game ends and gives payoffs of (-1,-3) and (2,1) respectively.

This game differs from strategic form game as, among others, decisions are not taken simultaneously but sequentially. Specifically, before choosing its strategy the incumbent observes the move of the entrant. In such case the incumbent can condition

Figure 7.1: Entry game.

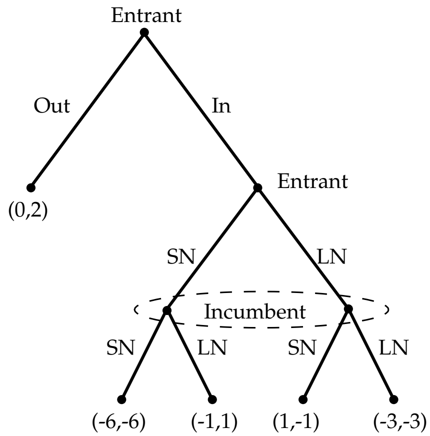


its decision on the observed decision of the opponent. The second stage of the game in which incumbent takes her decision is called a **subgame** (see part of the decision tree in the dashed line rectangle). Hence the entrant has two strategies (In, Out) equivalent to its decisions, but the incumbent has two strategies conditioned on observed moves: (F if entry occurs, A if entry occurs).

Formally speaking a (pure) **strategy** is a function mapping player's information to his action set. There are two Nash equilibria of this game: (In, A if entry occurs) and (Out, F if entry occurs). In both, players cannot increase their payoff by unilateral deviation. Interestingly the second Nash equilibrium is somehow inconsistent as it is based on the **empty threat**. Specifically the entrant stays out of the market because he is afraid of getting -1, when the entry occurs and the incumbent fights. Observe, however, that if the entry actually occurs the incumbent is better off by accommodating. Hence, the Nash equilibrium (Out, F if entry occurs) is not a subgame perfect Nash equilibrium, as the decision F, if the entry occurs, is not an equilibrium of the subgame (which is a single player decision problem in our simple example). More formally a **subgame perfect Nash equilibrium** (or SPNE for short) is the Nash equilibrium, such that for all subgames, the part of a strategy profile restricted to this subgame is a Nash equilibrium of this subgame. In finite games (i.e. games with finite number of periods), there is a simple way to find SPNE, namely using backward induction. Specifically consider the final subgame of our game and find the Nash equilibrium profile. In the game 7.1 it is A. Then choose the Nash equilibrium of the proceeding subgame, assuming that players know that in the following subgames a Nash equilibrium profile will be played. One finds a SPNE repeating this procedure to the initial node.

Extensive form games can also incorporate simultaneous moves. The example is presented in figure 7.2. The dashed ellipse denotes an **information set** of incumbent player, i.e. although incumbent observes if the entrant has entered or not, he does not observe however, whether entrant has chosen small or large niche (two nodes in the ellipse).

Figure 7.2: Niche choice game.



The post entry subgame can be alternatively expressed using the matrix notation in 7.6. Observe that a post entry subgame has two pure strategy Nash equilibria: (SN,LN) and (LN,SN). In the former one entrant gets -1, while in the latter 1. As a result we have two SPNE in the Niche choice game: (In, LN after entry, SN if firm E enters) and (Out, SN after entry, LN if firm E enters).

Table 7.6: Post entry (sub)game.

	SN	LN
SN	-6,-6	-1,1
LN	1,-1	-3,-3

To finish let us mention that a topic of strategic and extensive form games is much broader than covered at this level. In includes equilibrium refinements as well as analysis of auctions, Bayesian games, repeated strategic form games with celebrated Folks' theorem[4], or infinite horizon stochastic games.

[4]Folks' theorem states that if players are patient enough than any (convex combination of) individually rational payoff vectors of a strategic form game can by supported by a Nash equilibrium of infinitely repeated game.

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Chapter 8

Oligopoly and industrial organization

In this chapter we consider oligopoly markets, i.e. markets with a small number of competitors. The models of oligopolies describe a competition that is 'imperfect', in the sense that it may lead to an inefficient solution. One reason for such inefficient outcomes comes from the fact that competitors are not price takers and hence see an impact of own decisions (concerning prices or quantities produced e.g.) on other competitors behavior. A fundamental tool of such an analysis is the game theory. At the end of this chapter we also discuss a monopolistic competition model, that is important for international trade, growth theory and industrial organization.

There are various other topics covered in the industrial organization literature that we will not cover here. These include: formal analysis of horizontal and vertical boundaries of firms, make-or-buy decisions, incomplete contracting, dynamic pricing rivalry, mergers and acquisitions, entry and exit, research and development, and many others. Here we refer an interested reader to some textbooks: Tirole (1988), Church and Ware (2000), Besanko, Dranove, Shanley, and Schaefer (2007) or Pepall, Richards, and Norman (2008).

Before proceeding we introduce two common measures of market concentration. Suppose there are m firms in a market, each with a market share of s_i. Without loss of generality suppose that firms names are ordered by its market share, with $i = 1$ a market leader. Then $CR_n = \sum_{i=1}^{n} s_i$ is the **n-firm concentration ratio**. Typically one measures CR_4, that is a sum of market shares of 4 largest firms. The second typical measure is a **Herfindahl-Hirschman Index** (HHI for short): $HHI = \sum_{i=1}^{m} s_i^2$, which is a sum of squared market shares of all companies on the market. If there is a single company on the market $HHI = 1$. If there are m equal sized companies on the market $HHI = \frac{1}{m}$. The closer the HHI or CR_n to 1 the more concentrated is the market.

We now proceed to present four canonical models of oligopolistic markets.

8.1 Cournot model

We start with a Cournot model of oligopolistic competition. Consider $n = 2$ firms producing the homogeneous product each with constant marginal costs $c_i = c \geq 0$. The inverse demand for the product is given by $P(Q) = a - bQ$, where $a > c, b > 0$. Companies compete simultaneously choosing their production levels $q_i \in [0, \infty)$. The total supply $Q = q_1 + q_2$. The profit of a company choosing production level q_i when its competitor chooses q_{-i} is given by:

$$\pi_i(q_i, q_{-i}) = (P(q_i + q_{-i}) - c)q_i = (a - b(q_i + q_{-i}) - c)q_i.$$

The Cournot model can be analyzed using a game theoretical language, with each firm being a player with payoff function π_i and strategy set $A_i = [0, \infty)$. We now proceed to describe the Nash equilibrium of the Cournot game. Recall that the Nash equilibrium in this case is a pair of production levels q_1^*, q_2^* such that $\forall i$ and $\forall q_i \in [0, \infty)$ we have:

$$\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*),$$

that is the production profile such that no company can strictly increase its profits by unilateral (production) deviation. Observe that the first order conditions for interior production level q_i^{BR} solve $\frac{\partial \pi_i}{\partial q_i}(q_i^{BR}, q_{-i}) = 0$. With our assumption about marginal costs and demand we hence have:

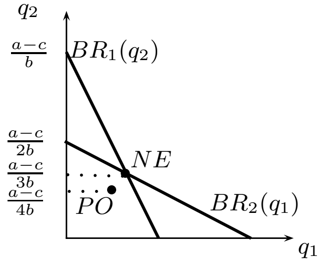
$$a - b(q_i^{BR} + q_{-i}) - c - bq_i^{BR} = 0,$$

which yields $q_i^{BR}(q_{-i}) = \frac{a-c-bq_{-i}}{2b}$. This is the best response production level of company i to the production level of company $-i$. Observe that the best response is a decreasing function of q_{-i}, i.e. the more competing firm is producing the less company i produce as a best response. In such a case we say that Cournot game exhibits **strategic substitutes**, i.e. company strategic choices are substitutes. To find the Nash equilibrium we solve a system of equations:

$$\begin{cases} q_1^{BR}(q_2^*) &=& q_1^*, \\ q_2^{BR}(q_1^*) &=& q_2^*. \end{cases}$$

Which under our assumptions gives a (unique) Nash equilibrium $(q_1^*, q_2^*) = (\frac{a-c}{3b}, \frac{a-c}{3b})$. The equilibrium total output level is $2\frac{a-c}{3b}$, while equilibrium price is $\frac{a+2c}{3}$. The Nash equilibrium profits are $\left(\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}\right)$. Figure 8.1 presents the two best response curves and Nash equilibrium.

Figure 8.1: Nash Equilibrium (NE) and Pareto optimal (PO) allocation in a Cournot game.

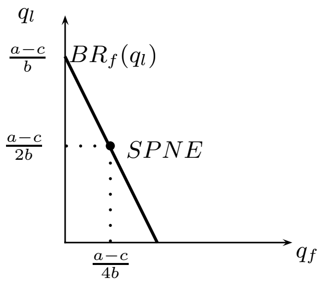


Interpreting, both companies produce the same level of output and none of them wants to unilaterally deviate from this output level. Moreover Cournot original analysis stressed stability of the Nash equilibrium, i.e. an iterative process $(q_i^t, q_{-i}^t)_{t=0}^{\infty}$ of quantity level updating $q_i^{t+1} = q_i^{BR}(q_{-i}^t)$ is convergent to the Nash equilibrium.

Interestingly the Nash equilibrium allocation is not Pareto optimal. To see that observe that both companies would be better off producing $(q_1^m, q_2^m) = (\frac{a-c}{4b}, \frac{a-c}{4b})$. Such allocation gives profits $\left(\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{8b}\right)$. The optimal total output $\frac{a-c}{2b} = \frac{a-c}{4b} + \frac{a-c}{4b}$ can be found by solving for optimal monopoly production level $\max_Q (a-bQ-c)Q$. Observe, however, that (q_1^m, q_2^m) is not a Nash equilibrium, i.e. even if companies coordinate to choose such production levels (e.g. creating a cartel) both of them have an incentive to deviate to $q_i^{BR}(\frac{a-c}{4b})$. It should be mentioned, however, that if the Cournot game is repeated infinitely many periods and companies are patient enough, the cooperative (cartel) solution can be supported as the Nash equilibrium of such dynamic game using appropriate punishment strategies.

The Cournot model analyzed in this section is the example as we assumed $m = 2$, homogeneous product, liner demand function P and constant (and equal across firms) marginal costs c. Generalizations are possible and analysis described in this section can be repeated similarly. However, one will not expect all of the properties of our example to remain valid. Specifically, when the inverse demand function is sufficiently convex, or when (differentiated) goods are complements then Cournot game may exhibit strategic complementarities, i.e. higher production of a competitor may lead a quantity increase in the best response of the other player. Moreover, the game may have multiple Nash equilibria, some of them stable/unstable (see Amir, 1996). Limiting properties of the Nash equilibrium allocation are also studied letting $n \to \infty$. Under some assumptions the Nash equilibrium price converge to marginal costs c, but it is not generally true. Also, it is not generally true that any Nash equilibrium price p_n is decreasing as a function of number of companies (see Amir and Lambson, 2000).

Figure 8.2: A subgame perfect Nash equilibrum (SPNE) of a Stackelberg game.



8.2 Stackelberg model

Another model of interest is a model of quantity leadership or Stackelberg. So consider again two firms producing a homogeneous product and deciding on own output $q_i \in [0, \infty)$. Both have constant marginal costs c but now one firm (market leader l) makes its choice first. Then (in the second stage) the follower f observes the output decision of the leader and then chooses its own output. Profits of each company are given by $\pi_i(q_l, q_f) = (a - b(q_l + q_f) - c)q_i$. We now analyze subgame perfect Nash equilibrium of the Stackelberg game. For this reason we first consider the follower choice for an observed choice of a leader q_l. Similarly as in the Cournot game we obtain the best response $q_f^{BR}(q_l) = \frac{a-c-bq_l}{2b}$. This is the optimal choice in the second stage, so let us now proceed to the first stage. Knowing the best response function of the follower, the leader decides on its optimal quantity choice solving: $\max_{q_l \geq 0} \pi_l(q_l, q_f^{BR}(q_l)) = (a - b(q_l + q_f^{BR}(q_l)) - c)q_l$. Observe that this objective depends only on q_l, i.e. best response of the leader is a single point. Solving we obtain $q_l^* = \frac{a-c}{2b}$ and $q_f^* = \frac{a-c}{4b}$. Total output is $\frac{3(a-c)}{4b}$ and price $\frac{a+3c}{4}$. Leader obtains profits: $\frac{(a-c)^2}{8b}$ and follower $\frac{(a-c)^2}{16b}$. Interpreting, the leader produces more than the follower and has higher profits. Also leader (resp. follower) produces more (resp. less) and has higher (resp. lower) profits than in the Nash equilibrium of the Cournot game. The Stackelberg output (resp. price) is higher (resp. lower) than in the Cournot game. See figure 8.2 for illustration or Amir and Grilo (1999) for more general cases.

8.3 Bertrand model

We now analyze Bertrand model of oligopolistic competition. Consider $m = 2$ firms producing a differentiated but substitutable products, each with constant marginal costs c_i. Demand for each product is given by $D_i(p_i, p_{-i}) = a - p_i + \gamma p_{-i}$, where $a > 0$ and $1 > \gamma > 0$ reflecting that products are substitutable. Firms compete simultaneously choosing their prices $p_i \in [0, \infty)$. When a pair of prices is chosen the profit of each firm is

$$\pi_i(p_i, p_{-i}) = (p_i - c_i)(a - p_i + \gamma p_{-i}).$$

We now find the Nash equilibrium of the Bertrand game. For this reason consider the first order conditions for optimal, interior price: $\frac{\partial \pi_i}{\partial p_i}(p_i^{BR}, p_{-i}) = 0$ which gives: $p_i^{BR}(p_{-i}) = \frac{a+c_i+\gamma p_{-i}}{2}$. Observe that as $\gamma > 0$ Bertrand model exhibits **strategic complementarities**: higher price of the rivals increase the best response price of a firm. The Nash equilibrium solves the system of equations:

$$\begin{cases} p_1^{BR}(p_2^*) &=& p_1^*, \\ p_2^{BR}(p_1^*) &=& p_2^*. \end{cases}$$

Which under our assumptions gives a (unique) Nash equilibrium with prices

$$p_i^* = \frac{(2+\gamma)a + 2c_i + \gamma c_{-i}}{4 - \gamma^2}.$$

Observe that the equilibrium price is increasing in own and rival's costs. Similarly to a Cournot model, it can be shown that Nash equilibrium allocation is inefficient. The Pareto optimal level of prices is higher than the Nash equilibrium one, though, itself is not a Nash equilibrium.

Again here we have focused on one example. For a detailed analysis of the Bertrand model we refer the reader to a book by Vives (2000). The extensions include non-linear demands, non-constant marginal costs, complementary products, productions constraints or use of mixed strategies (see Maskin (1986), Dasgupta and Maskin (1986), Dastidar (1995), Baye and Morgan (1999)). Also price leadership models (similar to Stackelberg) are considered (see Amir and Stepanova, 2006). We finish this section with an 'extreme' example of Bertnard competition with homogeneous products.

Example 8.1 (Bertrand with homogeneous products) *Similarly as above consider an example of $m = 2$ Bertrand competitors with constant marginal costs but homogeneous product with demand $D(p)$. Assume D is continuous, strictly decreasing whenever positive and there exists a price $\bar{p}$ such that $(\forall p \geq \bar{p})\, D(p) = 0$. If firms choose a price pair p_i, p_{-i} demand for firm's i output is:*

$$d_i(p_i, p_{-i}) = \begin{cases} 0 & if \quad p_i > p_{-i}, \\ \frac{D(p)}{2} & if \quad p_i = p_{-i}, \\ D(p) & if \quad p_i < p_{-i}. \end{cases}$$

Firms profits are hence: $\pi_i(p_i, p_{-i}) = (p_i - c)d_i(p_i, p_{-i})$. We claim there is a unique Nash equilibrium of this game with $(p_1, p_2) = (c, c)$. That is the unique Nash equilibrium prices are equal to marginal costs and companies have zero profits. To see *that (c, c) is indeed the Nash equilibrium observe that $(\forall p)\, \pi_i(p_i, c) \leq 0$. To see that this Nash equilibrium is unique, observe that $c > p_i \geq p_j$ cannot be an equilibrium as j-company is better off if it increases price to $p_j = c$. Similarly $p_i \geq p_j > c$ is also not a Nash equilibrium as the i-company can set price $p_j - \epsilon$ and increase its profits. To see that observe that $\lim_{\epsilon\to 0} \pi_i(p-\epsilon, p) = \lim_{\epsilon\to 0}(p-\epsilon-c)D(p-\epsilon) = (p-c)D(p) > (p-c)\frac{D(p)}{2} = \pi_i(p,p)$ and hence there exists an $\epsilon > 0$ such that $\pi_i(p-\epsilon, p) > \pi_i(p,p)$. So there is a price undercutting strategy to increase own profits. Interestingly, there is no least ϵ that accomplishes such a price undercut and hence firm i best response is not defined for $p_j > c$.*

The uniqueness of Nash equilibrium with prices equal to marginal costs is known as the Bertrand paradox, as there are only two companies in equilibrium and they set prices equal to marginal costs. This example suggests that Bertrand or price undercutting competition is severe, much more than Cournot. However, this is just a cooked example. We have seen that if the companies have slightly differentiated products or some production constraints (so that they cannot meet the whole demand on their own) the Bertrand paradox will not hold.

Finally let us mention that at the first glance the differences between Cournot and Bertrand may be misleading as in reality companies may choose both: either simultaneously set prices and quantities or sequentially, first compete in production possibilities (capital investment or scale of operations) (Cournot) and then, when production possibilities are fixed, compete in prices (Bertrand). Also there are markets in which in some business cycle phases companies compete a la Bertrand, while a la Cournot in the others. These issues are analyzed in a few papers including Kreps and Scheinkman (1983), Davidson and Deneckere (1986), Klemperer and Meyer (1989), or Kovenock, Deneckere, Faith, and Allen (2000) and d'Aspremont and Dos Santos Ferreira (2009) for more recent developments.

8.4 Monopolistic competition

The final model we analyze in this chapter is a model of monopolistic competition (also known as Chambarlain or Dixit and Stiglitz (1977) model). For this reason consider m (think of m as large) firms producing differentiated but substitutable products and (simultaneously) competing in prices. This is called a monopolistic competition as each firm is a monopolist in producing its own product, but as goods are (to some degree) substitutable firms must consider other prices as well. This is similar to Bertrand with differentiated good example we analyzed so far, but not quite, as with large m each firm will take a market price index P as given. Assume constant marginal costs c_i for each firm.

We consider an example of demand derived from CES utility function with n goods.

Recall (see example 2.8) that in such case $d_i(p_i, P) = \frac{1}{P} p_i^{\frac{1}{\rho-1}}$, where $P = \sum_{j=1}^m p_i^{\frac{\rho}{\rho-1}}$, $0 < \sigma < 1$ and we normalize consumer income $I = 1$. P is called a price index and as m is large we will assume that, when choosing its optimal price, each firm will take P as given.

For given CES demand functions d_i, a **monopolistic competitive equilibrium** is a vector of prices $(p_i^*)_{i=1}^n$ and index P^* such that:

1. $(\forall i)\, p_i^* \in \arg\max_{p_i} (p_i - c_i)(\frac{p_i^{\frac{1}{\rho-1}}}{P^*})$,
2. $P^* = \sum_{j=1}^n p_i^{*\,\frac{\rho}{\rho-1}}$.

Observe that this is not the Nash equilibrium of the corresponding Bertrand game as in the Bertrand game (even for large m) each firm should see its impact on the price index P. Formally such game is called **aggregative game** as each player plays against the aggregate and in equilibrium aggregate value is determined by joint actions of all players.

The monopolistic competitive equilibrium is easily characterized for CES preferences as the optimal price company i does not vary with P (but generally it does not need to be so). Specifically $p_i^* = \frac{c_i}{\rho}$ and observe price is, higher than marginal costs c_i as $\rho < 1$. Hence in equilibrium each firm has a margin resulting from its monopolistic power.

Finally for an aggregative game, if one wants the Nash equilibrium being equal to the aggregative equilibrium, one needs m to be large, usually a continuum. Hence economists also analyze large games with various concepts of equilibria (see Mas-Colell, 1984, Schmeidler, 1973, for seminal treatments).

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Chapter 9

General equilibrium

In chapter 4 we have introduced a concept of partial equilibrium, i.e. a pair of price of some commodity and quantity produced such that demand equals supply. We called such an equilibrium partial as we have taken demand and supply of a commodity of interest (say i) as given. Specifically, when deriving demand from a consumer maximization problem we have assumed that his/her income I is given and moreover, when defining partial equilibrium, we have taken demand and supply as correspondences mapping prices of this single commodity p_i, i.e. treating all other prices $p_{j \neq i}$ as parameters. This approach is useful if one analyzes a single market that is small relative to the others, i.e. when assumption, that situation on a given market is not influencing consumer income nor prices on other markets, is justified.

In this chapter we analyze the **general equilibrium**, i.e. an equilibrium in all markets simultaneously, where all markets clear and all prices are determined endogenously. This allows to identify interactions between markets. Our analysis will be organized around few questions:

- how to propose a meaningful definition of the general equilibrium, taking into account interactions between markets?
- what are the conditions under which such general equilibrium exists, i.e. is this likely that all markets will clear simultaneously?
- suppose by Ω we denote the set of all general equilibria of some economy. What are the conditions such that Ω is a singleton (there is a unique equilibrium)? Is the number of equilibria finite or infinite?
- what are the properties of a particular equilibrium? is equilibrium allocation efficient in some sense?

- is general equilibrium empirically meaningful, i.e. is it general enough to explain many real markets' situations? What testable restrictions an equilibrium imposes on the observed set of data?

Economists have studied these questions for a long time including contributions of Smith or Walras, but answered these questions relatively recently. The first formal treatment of these questions have been pursued by Arrow, Debreu and McKenzie (see Arrow and Debreu (1954) for example). The analysis conducted was positive in the sense, that they discussed whether an economy can be in equilibrium in all markets simultaneously, and what are the properties of such equilibrium, but were not aimed to argue if some real economy actually is in equilibrium or not.

We start with a simplified model of an **exchange economy**, where there is no production. It captures many phenomenas of our interest and allows to introduce the full model (with production) smoothly.

9.1 Exchange economy

We start by defining an economy and then its equilibrium. There are n consumers each with preferences over consumption set $X_i \subset \mathbb{R}^K_+$ represented by a utility function $u_i : X_i \to \mathbb{R}$. Each consumer has an initial endowment $e_i \in X_i$. We summarize economy by a list $E = (X_i, u_i, e_i)_{i=1}^n$. Vector x such that $x \in \times_{i=1}^n X_i$ is called an **allocation**, while allocation satisfying $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ is called **feasible**.

Definition 9.1 (Competitive equilibrium) *A competitive equilibrium of E is a pair of (p^*, x^*) such that $p \in \mathbb{R}^K_+$ and $x^* \in \times_{i=1}^n X_i$ and both consumer maximization (CM), and market clearing condition (MC) hold:*

(CM) $(\forall i)\, x_i^* \in \arg\max_{x_i \in X_i} u(x_i)$ *s.t.* $p^* \cdot x_i \leq p^* \cdot e_i$,

(MC) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n e_i$.

Few comments are in order. First, note that an equilibrium is a pair of vectors, i.e. notion of equilibrium captures the prices and quantity produced in all K-markets. Here we have assumed that prices are nonnegative but this could be generalized to some degree, if one wants to incorporate commodities that are 'bads'. Second, $p^* \cdot x_i$ is a scalar product of two vectors and $p^* \cdot x_i = \sum_{k=1}^K p_k^* x_{i,k}$. Third, consumer maximization condition (CM for short) means that each vector x_i^* belongs to the demand/supply correspondence of a consumer i. Specifically it requires that taking a vector of prices p^* as given, each x_i^* maximizes utility u_i subject to a budget constraint. This is the same maximization problem as we have considered in chapter 2, but now consumer income I is endogenous and equal to the value of endowment sold at market prices. Fourth, market clearing condition (MC for short) requires that demand equals supply. Although prices do not appear in this condition, this is the one that determines the equilibrium prices. MC is stated with equality sign but some authors state this condition with $\leq$ sign. This is an important difference as here we do not allow any endowment (think of garbage or pollution) not to be fully utilized[1]. Fifth, suppose an equilibrium p^*, x^* exists, then $\lambda p^*, q^*$ is an equilibrium as well, for any $\lambda > 0$. This is clear as the demand is homogeneous of degree zero in prices. This means that we have some degree of freedom in choosing prices[2]. Specifically we can normalize prices so that $\sum_{k=1}^{K} p_k = 1$ or such that $p_1 = 1$, where commodity 1 is called a numéraire. Also when we analyze number of equilibria, we mean number of normalized equilibria, i.e. with prices determined up to a scalar multiplication.

Condition MC can be stated as $\mathbf{0} \in z(p^*)$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}_+^K$ and

$$z(p) = \sum_{i=1}^{n} (x_i^*(p, p \cdot e_i) - e_i),$$

and x_i^* is a Marshallian demand as introduced in chapter 2. Correspondence z is called an **excess demand** as it expresses consumer demand net his/her endowment. If demand is single valued, MC requires vector $\mathbf{0} = z(p^*)$. We will assume that z is single valued for simplicity in our further analysis. An important property of z is so called **Warlas' Law**. Specifically, if consumers spend all their incomes, then for all $p \in \mathbb{R}_+^K$ we have $p \cdot z(p) = 0$, that is the value of excess demand is zero. Warlas' Law follows from linearity of prices and assumption that all consumers spend their whole budgets. It has important corollaries. It implies among other that, if $K - 1$ markets clear, then the last market must clear as well. Alternatively, if one claims that some market is not in equilibrium, he must point another market that is not in equilibrium as well.

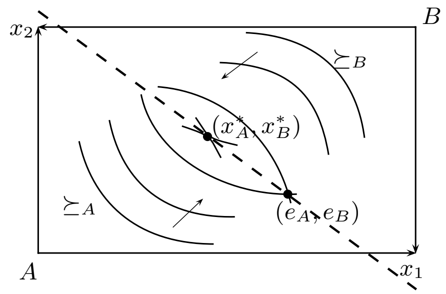
Exchange economy and its equilibrium for $K = 2$ and $n = 2$ can be nicely analyzed using **Edgeworth box**. We now show a graphical example and then continue with an algebraic one.

Example 9.1 *By* $1, 2$ *denote two goods and by* A, B *two consumers. Let endowment* $(e_A, e_B) = (e_{A,1}, e_{A,2}, e_{B,1}, e_{B,2})$ *and preferences* $\succeq_A, \succeq_B$ *be given. Consider a graph in figure 9.1. We put consumer A in the left, bottom corner while consumer B on the top, right. Each consumers' preferences are convex and increase towards the middle of the box (as indicated by arrows). The length of the horizontal axis is simply the total endowment of good 1, i.e.* $e_{A,1} + e_{B,1}$. *Similarly the length of the vertical axis is the total endowment of good 2, and given by* $e_{A,2} + e_{B,2}$. *A point* (e_A, e_B) *denotes, how*

[1] To see importance of this assumption refer to the paper by Cornet, Topuzu, and Yildiz (2003).

[2] Recall the (CM) condition. Note, that prices enter the optimization problem only via the budget constraint. However, for any $\lambda > 0$, $p^* \cdot x_i \leq p^* \cdot e_i$ if and only if $\lambda p^* \cdot x_i \leq \lambda p^* \cdot e_i$. Hence the value of prices does not matter as long as their relation is unchanged.

Figure 9.1: Example of an Edgeworth box.



total endowment is divided between consumers in the economy. Allocation (e_A, e_B) is not Pareto optimal, however. Observe that we can find an allocation that gives both consumers higher utility: (x_A^*, x_B^*). To reach this allocation from an initial endowment (e_A, e_B) it is sufficient that each consumer optimizes his utility subject to a budget constraint given by the dashed line. A pair of prices (given by the slope of the dashed line) and allocation (x_A^*, x_B^*) constitutes a competitive equilibrium in this economy.

Example 9.2 *Consider two consumers $i = A, B$ with preferences over two goods $k = 1, 2$. Let $u_A(x_{A,1}, x_{A,2}) = x_{A,1}^{\alpha} x_{A,2}^{1-\alpha}$ and $u_B(x_{B,1}, x_{B,2}) = x_{B,1}^{\beta} x_{B,2}^{1-\beta}$. Also let $e_A = (1, 0)$ and $e_B = (0, 1)$. In such a case demand is given by (recall example 2.7):*

$$x_A^* = \begin{bmatrix} \frac{\alpha(1p_1+0p_2)}{p_1} \\ \frac{(1-\alpha)(1p_1+0p_2)}{p_2} \end{bmatrix}, \quad x_B^* = \begin{bmatrix} \frac{\beta(0p_1+1p_2)}{p_1} \\ \frac{(1-\beta)(0p_1+1p_2)}{p_2} \end{bmatrix}.$$

Equating aggregate demand to supply we have:

$$x_A^* + x_B^* = \begin{bmatrix} x_{A,1}^* + x_{B,1}^* \\ x_{A,2}^* + x_{B,2}^* \end{bmatrix} = \begin{bmatrix} \alpha + \frac{\beta p_2^*}{p_1^*} \\ \frac{(1-\alpha)p_1^*}{p_2^*} + 1 - \beta \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

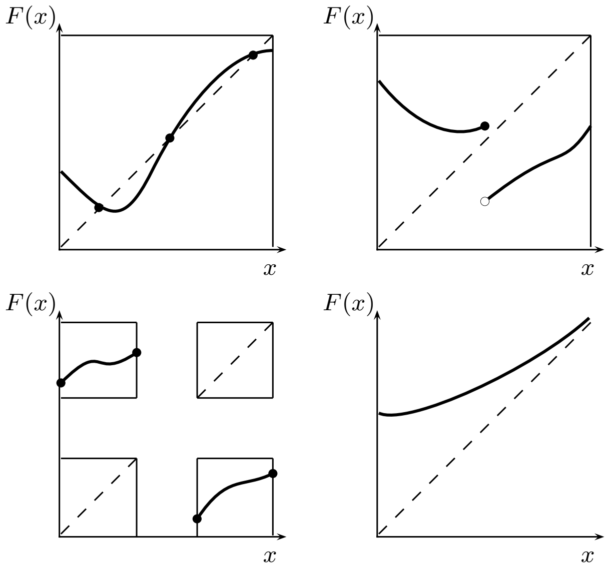
Which gives[3] $\frac{p_2^*}{p_1^*} = \frac{1-\alpha}{\beta}$.

Having defined an equilibrium the first question that checks consistency of this concept is, under what conditions the equilibrium exists. We will not address this question in details here, but we stress that to prove the equilibrium existence it is sufficient to find a vector p^* such that $\mathbf{0} \in z(p^*)$. Equivalently one can fixed a fixed point of a map F mapping set $\triangle = \{p : \mathbb{R}_+^K : \sum_k p_k = 1\}$ into itself, and defined by:

$$F_k(p) = \frac{p_k + \max\{0, \tilde{z}_k(p)\}}{1 + \sum_s \max\{0, \tilde{z}_s(p)\}},$$

[3] Observe again, that equilibrium prices are determined via their relation. Clearly, for any $\lambda > 0$, $\lambda p_1^*, \lambda p_2^*$ is also equilibrium price, since $\frac{\lambda p_2^*}{\lambda p_1^*} = \frac{p_2^*}{p_1^*} = \frac{1-\alpha}{\beta}$.

Figure 9.2: Brower fixed point argument.



where $\tilde{z}$ is some function selected from z, or simply z if single valued. Recall that a fixed point of F is $p^* = F(p^*)$. To see that any fixed point of F is an equilibrium price vector observe, that $p = F(p)$ assures for all k: $\max\{0, \tilde{z}_k(p)\} = p_k \sum_s \max\{0, \tilde{z}_s(p)\}$. Multiplying by $\tilde{z}_k$ and summing over all k we have:

$$\sum_k \tilde{z}_k \max\{0, \tilde{z}_k(p)\} = \sum_s \max\{0, \tilde{z}_s(p)\} \sum_k z_k p_k = 0,$$

where the last equality follows by Walras' law. Hence we have: $\sum_k \tilde{z}_k \max\{0, \tilde{z}_k(p)\} = 0$, where each term in the sum is nonnegative. But as the sum is equal to zero hence $\tilde{z}_k \max\{0, \tilde{z}_k(p)\} = 0$ for all k implying $\tilde{z}_k(p) \leq 0$. To assure equality we must assume some additional property like desirability of goods whose prices equal zero. This can be done.

As a result to prove the equilibrium existence, it is sufficient to find a fixed point of F for some selection $\tilde{z}$. A **Brower fixed point theorem** assures that a continuous function mapping nonempty, compact and convex set into itself has a fixed point (see figure 9.2 to get some intuition, why each of these conditions is necessary). Hence it suffices to find a continuous function selected from z and appeal to this theorem. Summing up, conditions guaranteeing the equilibrium existence (for economies with finite number of commodities) are known and can be obtained under quite general settings. This is summarized here.

Theorem 9.1 *Assume each* $X_i = \mathbb{R}^K_+$*,* u_i *is strictly increasing, continuous and quasi-concave with* $(\forall i,k)\, e_{i,k} > 0$*. Then there exists a competitive equilibrium.*

Generally equilibria are not unique. However, interestingly almost all economies have a finite and odd number of them. Having established equilibrium existence we analyze its welfare properties. We start with a definition.

Definition 9.2 (Pareto optimality) *A feasible allocation* x *is Pareto optimal if there does not exist another feasible allocation* $\hat{x}$*, such that:*

- $(\forall i)\, u_i(\hat{x}_i) \geq u_i(x_i)$ *and*
- $(\exists i)\, u_i(\hat{x}_i) > u_i(x_i)$*.*

Few comments. First, Pareto optimality requires efficiency, i.e. there are no wasted resources. Second, one can expect there are many Pareto optimal allocations. Third, Pareto optimality will not be confused with any sort of 'fairness'. Having that we can state a celebrated first welfare theorem.

Theorem 9.2 (First welfare theorem) *Assume each* $X_i = \mathbb{R}^K_+$ *and* u_i *is strictly increasing. If* p^*, x^* *is a competitive equilibrium, then* x^* *is Pareto efficient.*

The first welfare theorem states that equilibrium allocation is not wasting resources, or that equilibrium allocations are on the Pareto frontier. The second welfare theorem asks if any allocation from a Pareto frontier can be supported as the equilibrium allocation.

Theorem 9.3 (Second welfare theorem) *Let* x^* *be a Pareto optimal allocation such that* $(\forall i,k)\, x^*_{i,k} > 0$*. Assume that each* u_i *is continuous, quasi-concave and strictly increasing. Then there exists a price vector* $p^* \in \mathbb{R}^k_+$*,* $p^* \neq 0$ *such that* x^*, p^* *is a competitive equilibrium of an economy with endowments* $e_i = x^*_i$*.*

Both welfare theorem nicely separate efficiency and welfare / redistribution problems. Observe that the second welfare theorem is not a direct converse to the first, as it requires preferences to be convex. Indeed for non-convex preferences it may be impossible to define linear prices, under which markets could clear. However even if preferences are non-convex but the number of consumers large, then one can conclude that any interior Pareto optimal allocation can be supported by liner prices (see Anderson, 1978).

To understand distinction between first and second welfare theorem, and also clarify the role of prices in equilibrium we consider the following **(welfare) maxi-**

mization problem:

$$\max_{x \in \times_{i=1}^{n} X_i} \sum_{i=1}^{n} \alpha_i u_i(x_i), \tag{9.1}$$
$$\text{s.t.} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i,$$

for some vector of (positive) consumers' weights $(\alpha_i)_{i=1}^{n}$. We now state a very important link between the set of Pareto optimal allocations, solutions to the welfare maximization problem and competitive equilibrium allocations.

Theorem 9.4 (Negishi (1960)) *Assume that each u_i is continuous, concave and strictly increasing.*

1. *Let x^* be a Pareto optimal allocation. Then there exists a vector of weights $(\alpha_i^*) \geq 0$ $\alpha^* \neq 0$, such that x^* solves the problem (9.1).*

2. *Let x^* solve the problem (9.1) for some weights (α_i^*) such that each $\alpha_i^* > 0$. Then x^* is Pareto optimal.*

3. *If x^*, p^* is a competitive equilibrium such that each $p^* \cdot x_i^* > 0$, then x^* solves the problem (9.1) for $\alpha_i = \frac{1}{\lambda_i}$ where λ_i is consumer's i marginal utility of income.*

4. *If $(\forall i, k)\, e_{i,k} > 0$ and x^* solves problem (9.1), then there exists a price vector p^* such that x^*, p^* is a competitive equilibrium. In this equilibrium $\alpha_i = \frac{1}{\lambda_i}$ is consumer's i marginal utility of income.*

The prices in this theorem are simply Lagrange multipliers associated with feasibility constraints. More on this interpretation of prices can be found in Bewley (2007) textbook.

Finally we consider the question, whether the general equilibrium analysis restricts observed data to some extent. This is addressed in the theorem of Brown and Matzkin (1996). So consider a scenario that an observer makes several observations of an economy. Each observation t consists of a price vector p^t, aggregate endowment e^t and income distribution $\{I_1^t, \ldots, I_n^t\}$. We say that this set of data for $t = 0, \ldots, T$ is **Walrasian rationalizable**, if there exists an increasing utility functions $u_i, i = 1, \ldots, n$ generating demand functions $x_i^*(p^t, I_i^t)$, such that $\sum_i x_i^*(p^t, I_i^t) = e^t$ for all $t = 0, \ldots, T$.

Theorem 9.5 (Brown and Matzkin (1996)) *Suppose such a set of observations is given. Then, there is an algorithm that could determine (in a finite number of steps), whether or not the set of observations is Walrasian rationalizable.*

Note: that not all observations are Walrasian rationalizable.

9.2 Extensions

Much of the analysis from the previous section can be easily generalized to allow for many extensions of the basic exchange economy model. We briefly present such three here: production economy, dynamic economy and economy with assets. More examples of general equilibrium analysis can be found in textbooks by Ellickson (1993), McKenzie (2002) and Bewley (2007).

Example 9.3 (Production economy) *Consider m firms each with production set $Y_j \subset \mathbb{R}^K$. Recall positive entries of a vector $y \in Y_j$ denote outputs, while negative denote inputs. By π_j we denote a profit of firm j. By $\theta_{i,j} \in [0,1]$ we denote a fraction of company j owned by a consumer i. Each consumer has utility $u_i : X_i \to \mathbb{R}_+$, endowment $e_i \in X_i$ and owns fraction of each firms profit $\theta_{i,1}, \ldots, \theta_{i,J}$. A competitive equilibrium is defined as a triple x^*, y^*, p^* and profits π^* such that $\forall i, j\, x_i^* \in X_i$, $y_j^* \in Y_j$ and $p^* \in \mathbb{R}_+^K$ such that:*

(CM) $(\forall i)\, x_i^* \in \arg\max_{x_i \in X_i} u(x_i)$ *s.t.* $p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^m \theta_{i,j}\pi_j^*$,

(FM) $(\forall j) p^* \cdot y_j^* \geq p^* \cdot y_j$ *for all* $y_j \in Y_j$ *and we set* $\pi_j^* = p^* \cdot y_j^*$,

(MC) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n e_i + \sum_{j=1}^m y_j^*$.

The new thing in this definition is the the second condition (firm maximization or FM for short). This requires that each firm chooses a bundle that is a profit maximizing. In equilibrium each consumer takes both prices and firms' profit as given. Existence of competitive equilibrium for a production economy can be similarly established for closed, convex Y_j containing $\mathbf{0}$. Can competitive equilibrium exist with increasing returns to scale production function? Moreover definition of Pareto optimality is simply extended: a feasible allocation x, y is Pareto optimal, if there is no other feasible allocation $\hat{x}, \hat{y}$ such that condition (9.2) holds. Also both the first and the second welfare theorem hold, the latter for convex Y_j.

The general equilibrium analysis can be also easily extended to allow specific dynamic context. Specifically, observe that in the analysis so far we can interpret goods, as goods in separate (finite number of) moments in time. However, more insight about dynamic economies can be obtained, when one incorporates time explicitly into preferences and production possibilities. This is done in the next example.

Example 9.4 (Equilibrium in time) *Consider a $T+1$-period, production economy with one consumer with preferences over consumption streams $c_0, c_1 \ldots, c_T$ given by: $\sum_{t=0}^T \beta^t u(c_t)$ where $0 < \beta < 1$. Each period consumer owns a unit of time and in period $t = 0$ has also an endowment of capital $k_0 > 0$. Consumer can accumulate own capital using transition: $k_{t+1} = (1-\delta)k_t + i_t$, where $i_t \in \mathbb{R}$ stands for investment* *and $\delta \in [0,1]$ for depreciation. There is a single firm with constant returns to scale production function $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$, producing $y_t = f(k_t, l_t)$. A feasible allocation in this economy is a list of $\{c_t, l_t, k_t, i_t, y_t\}_{t=0}^T$ such that $c_t + i_t = y_t = f(k_t, l_t)$ and $k_{t+1} = (1-\delta)k_t + i_t$, $k_t \geq 0$, $l_t \in [0,1]$. Note that once a sequence of $\{k_t, l_t\}$ have been chosen sequences $\{y_t\}$ and $\{i_t\}$ are determined. Hence we can simplify notation dropping these two sequences.*

A competitive equilibrium of such economy is a list $\{c_t^*, l_t^*, k_t^*, l_t^f, k_t^f\}_{t=0}^T$ and prices $\{p_t^*, w_t^*, r_t^*\}_{t=0}^T$ such that:

(CM) $\{c_t^*, l_t^*, k_t^*\}_{t=0}^T$ *solves*

$$\max_{\{c_t, l_t, k_t\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \;\; s.t.$$
$$\sum_{t=0}^T p_t^*(c_t + k_{t+1} - k_t(1-\delta)) \leq \sum_{t=0}^T (w_t^* l_t + r_t^* k_t),$$
$$(\forall t)\, k_t \geq 0, l_t \in [0,1], \;\; \text{and } k_0 \text{ is given},$$

(FM) $\{l_t^f, k_t^f\}_{t=0}^T$ *solves:*

$$\max_{\{k_t, l_t\}_{t=0}^T} \sum_{t=0}^T (p_t^* f(k_t, l_t) - w_t^* l_t - r_t^* k_t),$$

(CM) $(\forall t)\, c_t^* + k_{t+1}^* - (1-\delta)k_t^* = f(k_t^*, l_t^*)$, $l_t^* = l_t^f$, $k_t^* = k_t^f$.

A few comments concerning this definition. First, the consumer has a single budget constraint. He earns selling his labor services and renting capital. The amount earned is used to cover consumption and investment spendings. Second, the firm's problem is dynamic but there is not explicit discounting. This means that equilibrium prices p_t^* must incorporate time preferences. Third, as (in our case) production function does not depend on past decisions and also as the firm does not own any capital, we can alternatively write FM as:

$$(\forall t)\, k_t^f, l_t^f \;\; \text{solves} \;\; \max_{k_t, l_t \geq 0} p_t^* f(k_t, l_t) - w_t^* l_t - r_t^* k_t.$$

*Fourth, as the technology is constant returns to scale (and implies zero profit condition) we do not need to incorporate profits into household budget constraint. Fifth, markets clear every period, i.e. for every good separately. As a result the consumption in every period has a separate price p_t^*. Sixth, as the consumer has a single budget constraint he can (potentially) freely transfer the income between periods. This, accompanied with other comments, leads us to interpret markets in this definition as **future markets** (for date-contingent claims) being opened only in period $t = 0$. As a* *result this definition is sometimes referred to as* ***Arrow-Debreu competitive equilibrium****, while in the case, where consumer has a sequence of budget constraints, we say about* ***sequential competitive equilibrium****.*

More on equilibrium in time can be found in Stokey, Lucas, and Prescott (1989), chapter 15. Finally we discuss an example of an asset economy.

Example 9.5 (Asset economy) *Consider a two period economy, with K goods and S states of the world. There is no production. Consumers consume only in the second period and have preferences $\sum_{s=1}^{S} \pi_{i,s} u_i(x_s)$, where $\pi_{i,s}$ is a vector of (perhaps subjective) probabilities and $u_i : \mathbb{R}_+^K \to \mathbb{R}$. The timing is the following: in the first period s in unknown and trade (exchange) takes place, in the second period state s is revealed to all consumers, contracts are fulfilled and consumption takes place. Consumers can condition their consumption choices on the realized state of the world, hence the consumption set is $X_i = \mathbb{R}_+^{KS}$. Consumers have endowments $e_i \in X_i$ denoting a vector of state dependent endowments of physical commodities in $\mathbb{R}_+^K$. An allocation $x^* = (x_1, \ldots, x_n) \in \times_{i=1}^n X_i$ is feasible if $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$. Observe this requires that each coordinate of both vectors is equal, i.e. markets clear in every state for every physical commodity. An* ***(Arrow-Debreu) competitive equilibrium*** *is a pair x^*, p^* such that, $x^* \in \times_{i=1}^n X_i$ and $p^* \in \mathbb{R}^{KS}$ and:*

(CM) $(\forall i)\, x_i^* \in \arg\max_{x_i \in X_i} \sum_{s=1}^{S} \pi_{i,s} u_i(x_{i,s})$ *s.t.* $p^* \cdot x_i \leq p^* \cdot e_i$,

(MC) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n e_i$.

Few comments concerning this definition. First observe that consumers maximize expected utility choosing state contingent consumption vectors. Again there is a single budget constraint, indicating that trade is in future (state-contingent) assets. Such assets are called ***Arrow securities****, and they give a vector of physical goods in state s and zero otherwise. We implicitly assume that there are markets for all Arrow securities (this implies that markets are* ***complete****). There is an alternative definition of a competitive equilibrium (called* ***Radner equilibrium****) explicitly incorporating asset (other than Arrow securities) markets and their structures. If assets do not allow to span the whole uncertainty space, we say that markets are* ***incomplete****.*

Detailed economic analysis of equilibrium with asset markets and uncertainty including complete and incomplete market case can be found in excellent textbooks by Werner and LeRoy (2001) and Magill and Quinzii (1996).

We finish with mentioning some current research in the general equilibrium. Apart from the already mentioned (dynamic economies, incomplete markets or both) this includes work on equilibrum with infinitely many goods or consumers (Aliprantis, Brown, and Burkinshaw, 1990), general equilibrium with adverse selection (Rustichini and Siconolfi, 2008), moral hazard (Jerez, 2005), as well as endogenous risk (Magill and

Quinzii, 2009) and extensive work on recursive competitive equilibrium (Ljungqvist and Sargent, 2000, part III). See also Ginsburgh and Keyzer (1997) for an exposition of computable and applied general equilibrium theory.

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Chapter 10

Asymmetric information

In this chapter we sketch two important problems of informational economics, that of: adverse selection and principal agent models. Both incorporate assumption that information is asymmetrically distributed between the parties. Adverse selection concerns the situation, when information is asymmetric before signing of the contract, while principal-agent models concern the situation, where asymmetric information develops subsequently during the course of interactions. Such asymmetric information can result for example from actions taken by one side.

10.1 Adverse selection

Consider three examples:

- on the car insurance market, a driver typically knows more about its driving skills than an insurance company,
- on the second-hand sale market, a seller usually knows more about a product quality than a potential purchaser,
- a worker usually knows more about its personal skills than its employer.

In all these cases information is asymmetric and can adversely impact the uninformed side. Hence the uninformed side should take these considerations into account. We now (following Mas-Colell, Whinston, and Green (1995)) consider the third example but the same logic applies to other.

Consider the labor market with many firms each with a constant returns to scale production function with a single input (labor). Firms are risk neutral and price takers. Price of a product is normalized to 1. Firms hire workers. Workers differ in terms of their productivity $\theta \in [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. The distribution of productivities is given by f. Workers obtain salary w, if work but $r(\theta)$ of their reservation salary, if stay at home. If firms cannot observe individual workers' productivities, the competitive equilibrium is a pair w^*, Θ^* such that:

$$\begin{aligned} w^* &= E[\theta : \theta \in \Theta^*], \\ \Theta^* &= \{\theta \in [\underline{\theta}, \overline{\theta}] : r(\theta) \leq w^*\}. \end{aligned}$$

The first condition equals wage with average productivity of all workers that (second condition) accept such wage. This restriction is called the adverse selection restriction, i.e. firms properly anticipate that, when salary is too small the high productive workers will stay out of the market (and enjoy $r(\theta)$).

This condition may lead to nonexistence of equilibrium, its multiplicity or even the total collapse of the market, e.g. if the firms expects only the least productive workers to stay at the market and offer corresponding wage. All of these can occur even if $r(\theta) \geq \theta$, i.e. it is efficient to employ all workers if information was public.

Hence the equilibrium allocation does not need to be Pareto optimal and hence the conclusion of the first welfare theorem fails. Examining the assumptions of the first welfare theorem we immediately see that the inefficiency results, as we have a single market (for work) for different goods θ. For the efficiency of allocation, one would require that each productivity θ is traded in a separate market. If this is not possible (because of asymmetric information) agents are enforced to trade different goods under an average price. As a result the most efficient agents are not willing to work under such average price as this is to small to compensate their skills. They stay out of the market and hence the average is reduced and new group of agents go out. This process continue until equilibrium conditions are met. **Adverse selection** is hence said to occur if the trading decision of one (informed) party adversely influence the uninformed party.

Observe that, in the case of adverse selection, Pareto-optimality concept (used in the first welfare theorem) is somehow inappropriate. Specifically, it is not appealing to compare market allocations (with uninformed agents) to efficient allocations (allowing for perfect information). Hence the concept of constrained Pareto-optimality is introduced, where the Pareto-improvement is a subject to the same information constraints as specified in the model. Importantly: some competitive equilibria under adverse selection may be constrained Pareto-optimal.

Various solutions to the adverse selection problem (Akerlof, 1970) has been proposed including signaling and screening, both formalized using game theoretic concepts (see Rothschild and Stiglitz, 1976, Spence, 1973). **Signaling** is a class of games, where the informed party moves first and signals its private information. **Screening** is a class of games, where the uninformed party moves first and tries to screen the informed one for its private information. Equilibria in such games may be **separating** (separating workers with respect to their productivities) or **pooling** (not separating). Separating equilibria allocations may be constrained Pareto optimal in the case of signalling. In the case of screening usually pooling equilibria do not exists, and separating may be non-existent as well.

10.2 Principal-agent problem

In this section we consider the problem of principal rewarding agent for his work. Specifically an agent chooses an action $a \in \{\underline{a}, \overline{a}\}$ that affects a probability distribution f of realization of random output $y \in \{y_1, \dots, y_n\}$. The principal owns (random) production technology and contemplates an optimal reward contract for the agent. Principal is risk neutral, while risk-averse agent has (quasilinear) utility of income $u(w)$ and cost of action $c(a)$. Principal does not observe the effort (action) of the agent but only the realization of y. The problem of the principal is to find the optimal contract specifying wage as a function of income, or $\{w_i\}_{i=1}^n$ for short. Consider a problem of:

$$\max_{\{w_i\}_{i=1}^n, a^*} \sum_{i=1}^n (y_i - w_i) f(y_i | a^*) \text{ s.t.}$$

(PC) $\sum_{i=1}^n u(w_i) f(y_i | a^*) - c(a^*) \geq \bar{u}$,

(IC) $a^* \in \arg\max_{a \in A} \sum_{i=1}^n u(w_i) f(y_i | a) - c(a)$.

The principal wants to maximize an expected value of output y minus salary w. The first constraint is called **participation constraint** (or individual rationality) and reflects the outside option $\bar{u}$ of the agent, while the second is called **incentive compatibility** and assures that agent in motivated to choose the action a^* that principal is trying to implement. The principal must take both of these into account when choosing optimal w. The above problem can be solved in two steps, first the principal chooses the optimal contract $\{w_i\}$ implementing each action, and then compares profits for all actions $a \in A$. The latter is simple, hence we will focus on the former. Suppose that principal wants to implement $\overline{a}$ and he writes a Lagrangean:

$$\begin{aligned}
&\sum_{i=1}^n (y_i - w_i) f(y_i | \overline{a}) + \\
&\lambda \left[\sum_{i=1}^n u(w_i) f(y_i | \overline{a}) - c(\overline{a}) - \bar{u} \right] + \\
&\mu \left[\sum_{i=1}^n u(w_i) f(y_i | \overline{a}) - c(\overline{a}) - \sum_{i=1}^n u(w_i) f(y_i | \underline{a}) + c(\underline{a}) \right].
\end{aligned}$$

Multiplier λ is associated with participation constraint and μ with incentive compatibility. For differentiable primitives, the first order condition implies for all i:

$$-f(y_i|\overline{a}) + \lambda u'(w_i) f(y_i|\overline{a}) + \mu u'(w_i)[f(y_i|\overline{a}) - f(y_i|\underline{a})] = 0,$$

and hence:

$$\frac{1}{u'(w_i)} = \lambda + \mu \left[1 - \frac{f(y_i|\underline{a})}{f(y_i|\overline{a})} \right].$$

This is a fundamental first order condition for optimal (interior) contract $(w_i)_{i=1}^n$. It states that optimal contract should give inverse marginal utility equal to the sum of Lagrange multiplier for participation constraint (corresponding to the "fixed salary") and product of a Lagrange multiplier of incentive compatibility and one minus likelihood ratio $\frac{f(y_i|\underline{a})}{f(y_i|\overline{a})}$ (corresponding to the "variable pay“ or "motivating salary“). The likelihood ratio $\frac{f(y_i|\underline{a})}{f(y_i|\overline{a})}$ specifies how more/less likely it is to get a draw y_i, if action $\underline{a}$ is chosen relative to $\overline{a}$.

Usually participation constraint is binding hence $\lambda > 0$, while incentive compatibility may be binding or not. In the case of $\mu = 0$, we have $\forall i \frac{1}{u'(w_i)} = \lambda$ and hence salary w_i is independent of y_i. This indicates that optimal salary is fixed and fully insures an agent from the company's random result. On the other hand, if $\mu > 0$ then optimal salary is variable and follows the likelihood ratio $\frac{f(y_i|\underline{a})}{f(y_i|\overline{a})}$. Specifically if $y_i \to \frac{f(y_i|\underline{a})}{f(y_i|\overline{a})}$ is decreasing, then optimal contract w_i is increasing in y_i for decreasing marginal utility.

Finally observe that the Lagrangean is not symmetric in $c(\overline{a})$ and $c(\underline{a})$. Specifically $c(\overline{a})$ appears in both participating constraint and incentive compatibility, while $c(\overline{a})$ is present only in the latter. This implies that it is weakly better for the principal to motivate (decrease costs of preferred action), than punish (increase costs of alternative action).

The basic model (Grossman and Hart, 1983, Holmstrom and Milgrom, 1987, Rogerson, 1985) has been extended in both applied and theoretical research. These include Holmstrom and Milgrom (1991) analysis of reward with multiple tasking, optimal contract for a project work (Holmstrom, 1982), contracting under the career concerns (Gibbons and Murphy, 1992) or optimality of tournaments (Lazear and Rosen, 1981). More advanced studies include work with many principals or many agents, as well as more recent work in dynamic contracting (Spear and Srivastava, 1987) applied e.g. to design the optimal unemployment insurance (Hopenhayn and Nicolini, 1997) or model personal bankruptcy (see Ljungqvist and Sargent, 2000, part V).

Hidden action (or moral hazard) problems are closely related to **hidden information** problem, where an uninformed principal tries to extract information from an informed agent. That may be different from the adverse selection models analyzed before, where the information asymmetry was present even before contracting. Both kinds of problems are nicely analyzed in Salanie (1997) or Laffont and Martimort (2001).

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Chapter 11

Externalities and public goods

In this chapter we study a class of public goods and so called externalities. We skip exposition of many important social choice and political economy problems but give reference to Olson (1965), Sen (1970) or Persson and Tabellini (2000) instead.

11.1 Public goods

Consider the clasification of goods based on two characteristics: **excludability** and **rivalry**. One can, in principal, exclude some consumers from a consumption of a excludable good. The rivalry means that consumption of a good by one person reduces possibilities to consume this good by the others. **Private goods** are both excludable and rival. **Public goods** are goods that are nonexcludable and nonrival, while **club goods** are excludable but nonrival. The goods, we have considered so far, were private. The example of a public good is a lighthouse light, radio signal or knowledge.

We now consider the problem of some public good provision. We start by characterizing its efficient level and then move to the problem of its private provision. Consider two consumers with preferences given by $u_i(c_i, G)$ over private consumption c_i and a public good G. Assume each is endowed with income I_i, and that production of a public good is linear $G = g_1 + g_2$, where g_i is consumer i provision to the public good. Consider the following welfare maximization problem:

$$\max_{g_1, g_2} \alpha_1 u_1(I_1 - g_1, g_1 + g_2) + \alpha_2 u_2(I_2 - g_2, g_1 + g_2).$$

Observe that the choice of g_1 is non only influencing utility of consumer 1 but also (directly) consumer 2. Such effect is called externality. Assuming differentiability, the first order condition for interior g_i gives:

$$\alpha_1 \frac{\partial u_1}{\partial G}(c_1, G) + \alpha_2 \frac{\partial u_2}{\partial G}(c_2, G) = \alpha_1 \frac{\partial u_1}{\partial c_1}(c_1, G) = \alpha_2 \frac{\partial u_2}{\partial c_2}(c_2, G).$$

Rearranging:

$$\sum_{i=1}^{2} MRS_i = \frac{\frac{\partial u_1}{\partial G}(c_1,G)}{\frac{\partial u_1}{\partial c_1}(c_1,G)} + \frac{\frac{\partial u_2}{\partial G}(c_2,G)}{\frac{\partial u_2}{\partial c_2}(c_2,G)} = 1 = MRT.$$

Interpreting, the efficiency condition equates the marginal rate of transformation with the sum of marginal rates of substitution for all consumers. The latter condition is new and reflects the public character of a public good. Consider the following example:

Example 11.1 (Efficient provision of public good) *Let $u_i(c_i, G) = \gamma_i \ln G + c_i$. Then efficiency condition requires: $\sum_{i=1}^{2} \frac{\gamma_i}{G} = 1$ giving $G = \gamma_1 + \gamma_2$ and feasibility $c_1 + c_2 + G = I_1 + I_2$.*

As we will show in the moment, private provision of public goods is rarely efficient. To analyze the private provision of a public good in an (partial) equilibrium framework we must incorporate some strategic motives into a definition. To see that consider the following strategic form game between two players each contributing g_i to $G = g_1 + g_2$. We now aim to solve for Nash equilibrium of this game by calculating best responses. So consider the problem of consumer 1:

$$\begin{aligned} \max_{c_1, g_1} \quad & u_1(c_1, g_1 + g_2), \\ & \text{s.t. } c_1 + g_1 = I_1, \\ & \text{and } g_1 \geq 0. \end{aligned}$$

Substitute $g_1 = G - g_2$ and consider:

$$\begin{aligned} \max_{c_1, G} \quad & u_1(c_1, G), \\ & \text{s.t. } c_1 + G = I_1 + g_2, \\ & \text{and } G \geq g_2. \end{aligned}$$

The second formulation indicates that the consumer 1 is effectively choosing the total of public good G, for each level of g_2. Apart from the inequality constraint this is a standard consumer maximization problem. So denote by $d_1(I)$ the demand for G by consumer 1 at income I, and add inquality constraint to see that: $G = \max\{d_1(I_1 + g_2), g_2\}$ or equivalently that a Nash equilibrium profile (g_1^*, g_2^*) satisfies:

$$\begin{aligned} g_1^* &= \max\{d_1(I_1 + g_2^*) - g_2^*, 0\}, \\ g_2^* &= \max\{d_2(I_2 + g_1^*) - g_1^*, 0\}. \end{aligned}$$

To see more continue example 11.1.

Example 11.2 (Private provision of public good) *As before consider two consumers with $u_i(c_i, G) = \gamma_i \ln G + c_i$ and assume that $\gamma_1 > \gamma_2$. Then $d_i(I) = \gamma_i$ and* *(typically for quasilinear utilities) is income independent. Now the condition for the Nash equilibrium requires:*

$$\begin{aligned} g_1^* &= \max\{\gamma_1 - g_2^*, 0\}, \\ g_2^* &= \max\{\gamma_2 - g_1^*, 0\}, \end{aligned}$$

and summing:

$$\max\{\gamma_1, g_2^*\} = G^* = \max\{\gamma_2, g_1^*\}.$$

With $\gamma_1 > \gamma_2$ it implies: $g_1^* = \gamma_1$ with $g_2^* = 0$. It means that player 1 is contributing the whole amount of G, while player 2 is ***free-riding****, i.e. benefiting from a public good, but not contributing. Level $G^* = \gamma_1$ is, of course, not efficient as we have seen before.*

Economists considered many **mechanisms**, that try to implement efficient allocations in the presence of public goods. These include: voting, Lindhal (allocation and prices), or Groves-Clarke mechanism among others. More on that can be found in Cornes and Sandler (1996) and Moore (2007).

11.2 Externalities

The public goods analyzed in the previous chapter are only a special case of more general phenomena called externelities. By **externalities** we mean that an action of one decision maker influences directly (i.e. not via prices in the general equilibrium context) objective function of the other. The examples include the consumption or production externalities and both can cause positive or negative effects. The example of negative consumption externalities is cigarette consumption, while negative production externalities is pollution caused during the production process. A typical example of a positive consumption externality is a public good.

In the presence of externalities the conclusion of the first welfare theorem generally does not hold. The main reason in that, there are goods / services (namely externalities) that are not priced. Remember that the implicit assumption of the first welfare theorem was that all goods / services influencing utilities were priced. Now using a particular (production externality) example we address the problem and formulate some solutions.

Example 11.3 (Private and external costs) *Consider two firms: 1,2. One produces x using technology with (private) costs $c(x)$. There is also an external cost (pollution) of producing x namely $e(x)$ that is perceived by firm 2. Assume that c, x are monotone, convex and differentiable. The profits are:*

$$\begin{aligned} \pi_1(x) &= px - c(x), \\ \pi_2(x) &= -e(x). \end{aligned}$$

The competitive equilibrium requires $p = c'(x^{PC})$, *which gives too large production level as compared to the choice that maximizes the social welfare* $\pi_1(x) + \pi_2(x) = px - c(x) - e(x)$. *To see that consider the optimality condition* $p = c'(x^{PO}) + e'(x^{PO})$ *and observe that price should be equal to the total social costs including private and external effects.*

Economists proposed two main solutions to the above problem: taxes (Pigou) and opening of the missing markets (Coase). We discuss them in the next two examples.

Example 11.4 (Pigovian Tax) *Continue example 11.3 but now suppose that emission of external effects is taxed with a linear rate* t. *Now the problem of the first firm is:*

$$\max_x px - c(x) - tx,$$

and gives the first order condition $p = c'(x) + t$. *Now setting* $t = e'(x)$ *would restore efficiency of the allocation. Of course this solution is subject to the tax authority knowing function* e *and optimal level* x^{PO}.

Example 11.5 (Missing markets) *Continue example 11.3 but now suppose we introduce a market for externalities, where companies can trade (buy and sell) externalities (e.g. pollution), under the price* q. *In such a case, problems of the both companies are*

$$\max_{x_1} px_1 + qx_1 - c(x_1),$$
$$\max_{x_2} -qx_2 - e(x_2).$$

where the company 1 sells externalities and the company 2 buys them. In equilibrium we have $-e'(x_2) = q = c'(x_2) - p$. *When the market clears* $x_1 = x_2$ *and hence we obtain the efficient outcome. Observe that in equilibrium* $q < 0$ *indicating that externalities (pollution) is a 'bad'.*

The efficient solution obtained in 11.5 indicates that the problem of externalities is a problem of missing markets. When introducing a new market one should also specify endowments of property rights to the goods traded (of simply endowments). Coase argued that if trading is costless, it does not matter for the efficiency, who owns the property rights. In our example, indeed, it does not matter, if firm 1 has a right to pollute or firm 2 has a right for a clean air. Such allocation of property rights matters of course for a division of income / earnings among firms / consumers. Finally and from a different perspective, the missing markets argument can also illustrate that company two has an incentive to buy company 1 and coordinate production level to the optimal one. Indeed, as the joint profit will exceed the sum of individual profits, the company 2 will always have enough money to cover the market price (i.e. profit) of purchase of firm 1.

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