# On pure altruism

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**Abstract**

We propose a general framework for studying altruistic behaviour across a wide range of economic settings. In our approach, an agent (the parent) is represented by a function that maps the preferences of those they interact with (the descendant) into their own, thus capturing how one's objectives adjust in response to the preferences of others. Within this framework, we introduce and characterise a natural notion of *comparative altruism*: one parent is said to be more altruistic than another if the former's choices are always preferred by the descendant to the choices of the latter, for any possible preferences of the descendant. We use our insights to propose a definition of *absolute altruism* and show its equivalence to the parent obeying the *Pareto principle* when making decisions. Our results have immediate implications for various economic applications, including comparative statics in charitable giving and the problem of optimal choice of pocket-money in multi-dimensional consumption spaces.

**Keywords:** other-regarding preferences, altruism, comparative statics, comparative altruism, Pareto principle, charitable giving
**JEL Classification:** D1, D9, D11

# 1 Introduction

What does it mean to be an altruist? What does it mean to be *more* altruistic than someone else? How is other-regarding behaviour related to the egocentric preferences of the individual—those they would use when making decisions in isolation? We address

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these questions and shed new light on the nature of altruism. Rather than defining what altruism is, we draw our conclusions from two more fundamental notions: *comparative altruism* and a notion of *core preferences* of an individual. Our central insight is that altruism coincides with adherence to the Pareto principle: whenever the preferences of the decision-maker and those of the affected individual agree, an altruistic decision-maker always selects an alternative consistent with both. Equivalently, they depart from the wishes of either party only when a genuine conflict of interest arises.

To develop these insights, we adopt an approach that is deliberately as model–free as possible. We introduce it in Section 2. We focus on a stylised interaction in which a decision-maker (the parent) selects an alternative that affects both themselves and another individual (the descendant). Unlike the standard literature, we do not identify the decision-maker with a fixed preference relation. Instead, we represent them by a *preference function* mapping the preferences of the person they interact with into their own. Formally, an individual is represented by a function $A : \mathscr{P} \to \mathscr{P}$, where $\mathscr{P}$ denotes the set of preferences over a space $X$. For any $\succeq \in \mathscr{P}$, the preference relation $A(\succeq) \in \mathscr{P}$ describes the objective of the parent when they are interacting with a descendant endowed with $\succeq$.[1] This minimal structure allows us to study how one's objectives adjust as they interact with others whose preferences may differ from theirs, and makes it possible to concentrate on the fundamental behavioural features of altruism without committing to a particular utility–aggregation framework or welfare representation.

Although our analysis is general, our main focus is on a class of preference functions that we dub *anchored*. A preference function $A$ is anchored if there exists a preference relation $\trianglerighteq \in \mathscr{P}$ such that, for any preferences of the descendant $\succeq \in \mathscr{P}$ and any alternatives $x, x' \in X$ with $x' \sim x$, we have $x' \trianglerighteq x$ if, and only if, $x' A(\succeq) x$. Therefore, $\trianglerighteq$ represents stable preferences that the parent follows whenever doing so does not affect the descendant's well-being in any way. We interpret $\trianglerighteq$ as the parent's *core* preferences—those they would reveal in isolation, absent any concern for the descendant's welfare.[2] These preferences capture the parent's egocentric component.

With this structure in place, we introduce our notion of *comparative altruism* in Section 3. One individual is more altruistic than another if choices of the former are always more favourable to the descendant, regardless of their preferences. Our first main

[1] We assume that the parent has the exact knowledge regarding the beliefs of the descendant.

[2] In fact, any function $A$ admits at most one relation $\trianglerighteq$, as long as the domain $\mathscr{P}$ is sufficiently rich.

result (Theorem 1) shows that this behavioural notion is equivalent to the preference function of the more altruistic parent dominating that of the less altruistic one in the single crossing sense à la Spence-Mirrlees, induced by the preferences of the descendant. Hence the more altruistic parent is the one, whose objectives align more with the preferences of the descendant than those of the less altruistic one. Importantly, our insights rely only on choices and preferences, objects that are—at least in principle—observable.

Building on the notions of core preferences and comparative altruism, we define *absolute altruism* in Section 4. A decision-maker is altruistic if they are more altruistic than the paternalistic (egocentric) decision-maker who always adheres to their core preferences $\trianglerighteq$—the one represented by the constant preference function $A(\succeq) := \trianglerighteq$, for all $\succeq \in \mathscr{P}$. Combining these definitions yields our main conclusion (Theorem 2): an altruist is precisely a decision-maker who satisfies the *Pareto principle*; whenever the parent's core preferences coincide with the descendant's, the parent always follows them, irrespective of how the descendant behaves in situations where their preferences diverge. Equivalently, deviations occur only when the parent faces a genuine conflict of interest. This emerges naturally from our framework and provides a minimal yet powerful criterion for altruistic behaviour.[3] Our approach therefore delivers a unified and observable notion of altruism and comparative altruism that can be used broadly in economic applications.

The aforementioned interplay between the notions of absolute and comparative altruism uncovers a key structural property discussed in Theorem 3. When two altruistic parents share the same core preference $\trianglerighteq$, being more altruistic is equivalent to shifting farther away from these core preferences. Thus, comparative altruism can be interpreted in terms of the *cost* of accommodating others: for more altruistic individuals, the cost of adjusting their objectives toward the preferences of others is lower.

We conclude the paper in Section 5 with two applications. First, we investigate the relationship between altruism and charitable giving, as in Forsythe et al. (1994), Andreoni and Miller (2002), and Cox et al. (2008). We consider a setting in which the parent must divide money between themselves and the descendant, and provide general conditions under which more altruistic parents always choose higher donations. This result requires only minimal assumptions on the descendant's preferences which, in particular, need not be monotone and may include features such as inequality aversion. Thus, our definition

[3] This result connects our analysis to the social choice literature (see, e.g., Sen, 1970). After all, a preference function may be perceived in terms of preference aggregation.

of altruism is consistent with existing insights in the literature.

Our second application concerns optimal choice of pocket-money in a multi-dimensional consumption environment. In a standard setting with multi-dimensional bundles, the parent allocates income between their own consumption (private and/or public) and pocket money that the descendant may freely allocate to their own private consumption. Under some separability conditions, we show that more altruistic parents always provide more pocket money. To our knowledge, this application extends the scope of the altruism literature beyond one-dimensional consumption or strategy spaces, e.g., as in Cox et al. (2008), and illustrates how our framework can address richer economic problems.

**Related literature** Altruism is a topic studied thoroughly in Philosophy, Sociology, and Psychology, that has found its way to the Economics literature as soon as Adam Smith's *Theory of Moral Sentiments*. However, to the best of our knowledge, it entered formal economic analysis only in the 1970s with the contributions of Becker (1974, 1976), Collard (1975), and Kolm (1983). These works introduced altruism as *positive utility interdependence*, whereby an individual's utility depends on the outcomes or utilities of others. Following this tradition, we study how an altruistic decision-maker's choices relate to their concern for the well-being of the person they interact with.

Subsequent research has examined altruism and other-regarding preferences in diverse settings, including Koopmans (1960), Pearce (2008), Ray (1987), Kimball (1987), Bernheim and Stark (1988), Lindbeck and Weibull (1988), Bergstrom (1999), Hori (2009), Ray and Vohra (2020), and Vásquez and Weretka (2020, 2021). In this literature the utilities of others are embedded directly into the objective of the decision-maker. While analytically convenient, this approach ties altruism to unobservable cardinal representations that may affect behavioural predictions, making the problem ill-defined (see Remark 1 in the current paper). Our analysis relies solely on the ordinal structure of preferences and investigates how altruism can be identified from observable choices.

Axiomatic contributions, such as Saito (2015) and Galperti and Strulovici (2017), characterise specific models of altruism, focusing on what may be viewed as *ego-centric altruism*, where the decision-maker projects their own preferences onto others. This is suitable when agents lack information about others' preferences. Our perspective instead studies individuals who respond directly to (possibly different) preferences of those they interact with, and use it to identify altruistic behaviour.

Our framework is also related to the experimental literature on charitable giving and social preferences, beginning with Güth et al. (1982) and Forsythe et al. (1994) and including Andreoni (1989, 1990), Levine (1998), Andreoni and Miller (2002), List (2007), and Bellemare et al. (2008). This literature examines behaviour in dictator and ultimatum games and how donations vary with environment and beliefs. While our theory is not directly part of this strand, the environment we study resembles generalised dictator games. In fact, in one of our applications in Section 5, we return to charitable giving and obtain robust comparative statics using our notion of altruism.

The paper that is probably closest to ours is Cox et al. (2008), who conducts a comparative statics analysis of altruistic behaviour in the classical dictator and ultimatum game framework. In their model, individuals are represented by stable preferences over allocations of money kept versus donated, and altruism is captured through a single-crossing condition between the preferences of more and less altruistic types. Although the research question is similar, our approach emphasises the interaction between the decision-maker's actions and the welfare of the descendant, and is not restricted to a one-dimensional donation choice. Nevertheless, in our application to charitable giving, we recover the main comparative statics identified in Cox et al. (2008), showing that our notion of altruism is consistent with the one in this particular problem.[4]

It is also worth mentioning Dufwenberg et al. (2011), who study opportunity-based altruism in a general-equilibrium setting. In their framework, individuals care about others' outcomes as well as the budget sets available to them. Their focus is on welfare properties of equilibria, rather than on behavioural comparative statics of more versus less altruistic individuals, which is the central question of our paper.

A distinct line of work views altruism as an *emergent* rather than inherent phenomenon. Becker (1974, 1976), Hammond (1975), Kurz (1978), Simon (1993), and Bergstrom and Stark (1993) show how purely self-interested individuals may behave altruistically to sustain long-term cooperation or in order to survive natural selection in an evolutionary setting. Although related in theme, this literature—rooted in repeated and evolutionary games—does not address comparative statics of altruistic preferences or the observable, preference-based criteria central to our analysis.

[4] Cox et al. (2008) also discusses reciprocity, which is not considered in the current paper.

Although related, we abstract from reciprocity or inequality aversion, which require strategic, two-sided interactions. In our framework, the parent affects the descendant, but not vice versa, allowing us to isolate altruism toward the descendant's preferences without strategic motives. Models of reciprocity in Rabin (1993), Dufwenberg and Kirchsteiger (2004), Falk and Fischbacher (2006) and inequality aversion in Fehr and Schmidt (1999), Bolton and Ockenfels (2000) involve bilateral behaviour and are thus outside our scope. Extending our approach to two-sided interactions is left for future research.

# 2 Setup

We begin by introducing basic terminology and notation. Let $X$ represent the space of alternatives/actions/social states. By $\mathscr{P}$ we denote a set of preferences over $X$, i.e., reflexive, complete, and transitive binary relations. The set $\mathscr{P}$ may contain all possible preferences over $X$, or a subset thereof, e.g., measurable, (semi-)continuous, locally non-satiated, etc. We shall be explicit regarding $\mathscr{P}$ whenever required.

For any preference relation $\succeq \in \mathscr{P}$, we denote its symmetric and asymmetric parts by $\sim$ and $\succ$, respectively. By $\mathcal{I}$ we denote the global indifference relation over $X$, i.e., we have $x'\mathcal{I}x$, for all $x, x' \in X$. Finally, we denote the weak and strict lower-contour sets by $L_{\succeq}(x) := \big\{y \in X : y \preceq x\big\}$ and $L^{\circ}_{\succeq}(x) := \big\{y \in X : y \prec x\big\}$, respectively, for any $\succeq \in \mathscr{P}$ and $x \in X$. Similarly, upper-contour sets are given by $U_{\succeq}(x) := \big(L^{\circ}_{\succeq}(x)\big)^c$ and $U^{\circ}_{\succeq}(x) := \big(L_{\succeq}(x)\big)^c$, for any $\succeq \in \mathscr{P}$ and $x \in X$.

## 2.1 Preference functions

Throughout this paper we focus on a version of a dictatorship game between two agents: the *parent* $(P)$ and a *descendant* $(D)$. Knowing preferences of the descendant, the parent is responsible for choosing the alternative/social state $x \in X$ that will be consumed by both agents. In order to study altruism, we will allow for the objectives of the parent to change and adjust to preferences of the descendant.

Our main object of interest is a *preference function* $A : \mathscr{P} \to \mathscr{P}$ that maps the set $\mathscr{P}$ to itself. Here we interpret the value $A(\succeq) \in \mathscr{P}$ of the function $A$ as a preference/objective that the parent assumes when interacting with a descendant who is endowed with preferences $\succeq$. As a result, we identify the parent with the preference function. This captures how objectives of the parent change as they interact with others. However, as the preference function $A$ maps the space $\mathscr{P}$ to itself, we require that preferences of the parent belong to the same class as the descendant's.[5]

In some instances we will find it convenient to represent values of function $A$ with $\succeq_A := A(\succeq)$, with its symmetric and asymmetric counterparts denoted by $\sim_A$ and $\succ_A$.

## 2.2 Anchored preference functions

So far, we imposed no restrictions on values of the preference function $A$, allowing the parent to assume *any* preferences in $\mathscr{P}$ when interacting with the descendant. However, one may be interested in imposing a form of consistency between values of $A$, in order to reflect the *core preferences* of the parent, i.e., preferences that they would reveal in isolation when interacting with no other agent.

**Definition 1** (Anchored preference function)**.** A preference function $A : \mathscr{P} \to \mathscr{P}$ is *anchored* if there is $\trianglerighteq \in \mathscr{P}$ such that, for any $x, x' \in X$ and $\succeq \in \mathscr{P}$ with $x \sim x'$,

$$x' \trianglerighteq x \quad \text{if, and only if,} \quad x' A(\succeq)\, x.$$

The above notion imposes a form of consistency on values of the function $A$. Specifically, the definition captures the idea that the parent is endowed with some core preferences $\trianglerighteq$ when making the choice in isolation. However, whenever they interact with the descendant, their objectives may shift away from $\trianglerighteq$ to $A(\succeq)$. Nevertheless, whenever it does not affect the welfare of the descendant, the parent always breaks ties with the same relation $\trianglerighteq$. The next corollary follows directly from the definition.

**Corollary 1.** *If $\mathcal{I} \in \mathscr{P}$, then $A : \mathscr{P} \to \mathscr{P}$ is anchored to $\trianglerighteq \in \mathscr{P}$ only if $A(\mathcal{I}) = \trianglerighteq$.*

In general, a preference function $A$ need not be anchored to a unique preference relation $\trianglerighteq$. For example, suppose that $X = \{x, y, z\}$ and $\mathscr{P} = \{\succeq^1, \succeq^2, \succeq^3\}$, where

$$x \succ^1 y \succ^1 z, \quad y \succ^2 z \succ^2 x, \quad y \sim^3 z \succ^3 x.$$

Consider a constant function $A(\succeq) = \succeq^1$, for all $\succeq \in \mathscr{P}$. Clearly, it is anchored to $\succeq^1$, since it is a constant function. However, it is also anchored to $\succeq^2$. Indeed, the only time the descendant reveals indifference is if they are endowed with preferences $\succeq^3$.

[5] This is not critical to our analysis, but we consider this assumption to be desirable.

Specifically, we have $y \sim^3 z$. However, since $y \succ^1 z$ and $y \succ^2 z$, the function $A$ is anchored to $\succeq^1$ and $\succeq^2$. Therefore, it is not anchored to a unique $\trianglerighteq$ in $\mathscr{P}$. This is no longer the case once we assume that the space of preferences $\mathscr{P}$ is sufficiently rich.

**Definition 2** (Indifference-rich domain)**.** The domain $\mathscr{P}$ is *indifference-rich* if $x' \succeq x$ and $x \succeq' x'$, for some $\succeq, \succeq' \in \mathscr{P}$ and $x, x' \in X$, implies $x \sim'' x'$, for some $\succeq'' \in \mathscr{P}$.

Consider the following proposition.

**Proposition 1.** *Suppose that the space $\mathscr{P}$ is rich. A function $A : \mathscr{P} \to \mathscr{P}$ is anchored to some $\trianglerighteq, \trianglerighteq' \in \mathscr{P}$ only if $\trianglerighteq = \trianglerighteq'$.*

*Proof.* Suppose that $A$ is anchored to some $\trianglerighteq, \trianglerighteq'$ such that $\trianglerighteq \neq \trianglerighteq'$. Hence, there is some $x, x' \in X$ satisfying $x' \triangleright x$ and $x \trianglerighteq' x'$. Moreover, by assumption on the domain $\mathscr{P}$, there is some $\succeq \in \mathscr{P}$ such that $x \sim x'$. Since $A$ is anchored to $\trianglerighteq$, it must be that $x' \sim x$ and $x' \triangleright x$ implies $x' \succ_A x$, where $\succeq_A := A(\succeq)$. Similarly, since $A$ is anchored to $\trianglerighteq'$, it must be that $x \sim x'$ and $x \trianglerighteq x'$ implies $x A(\succeq) x'$, which yields a contradiction. □

Note that the richness condition in Proposition 1 was not satisfied in the previous example. Even though we had $x \succ^1 z$ and $z \succ^2 x$, there was no preference $\succeq \in \mathscr{P}$ that satisfied $x \sim z$. Clearly, any set $\mathscr{P}$ containing $\mathcal{I}$ satisfies the condition trivially. However, this is not necessary, as shown in the example below.

**Example 1.** Suppose that $X = \mathbb{R}^\ell_+$ and $\mathscr{P}$ consists of strictly increasing preferences over $X$. Hence, $\mathcal{I} \notin \mathscr{P}$. Let $x' \succeq x$ and $x \succeq' x'$, for some $\succeq, \succeq' \in \mathscr{P}$. If either $x' \sim x$ or $x \sim' x'$, we are done. If not, then $x$ and $x'$ must be unordered. Take any vector $p \in \mathbb{R}^\ell_{++}$ such that $p \cdot x = p \cdot x'$. Clearly, it exists. Define a preference relation $\succeq''$ by: $y' \succeq'' y$ if, and only if, $p \cdot y' \geq p \cdot y$, which is strictly increasing, while $x \sim'' x'$.

## 2.3 Examples of preference functions

To fix our ideas, in this subsection we present examples of preference functions.

**Example 2** (Conformism)**.** For any $\mathscr{P}$, we identify a *conformist* with the identity mapping $A(\succeq) = \succeq$, i.e., preferences of the parent always mimic the descendant's.

**Example 3** (Contrarianism)**.** Let $\mathscr{P}$ denote the space of all preferences over $X$. A *contrarian* function $A$ is given by: $x' A(\succeq) x$ if, and only if, $x \succeq x'$, for all $x, x' \in X$. Hence, their preferences are always opposite to the descendant's.

Notice that, unlike conformism, contrarianism need not be well-defined for any set of preferences $\mathscr{P}$. Indeed, it must be that, for any $\succeq \in \mathscr{P}$, the set $\mathscr{P}$ admits the opposite preference relation $\succeq'$, given by: $x' \succeq' x$ only if $x \succeq x$.

**Example 4** (Egocentrism/Paternalism)**.** We identify *egocentrism/paternalism* with a constant preference function $A$, i.e., there is some $\trianglerighteq \in \mathscr{P}$, such that $A(\succeq) = \trianglerighteq$, for all $\succeq \in \mathscr{P}$. Thus, an egocentric parent is never affected by the descendant's preferences.

Our next example is inspired by the social choice literature.

**Example 5** (Borda rule)**.** Suppose that $X$ is finite and $\mu$ is the uniform probability measure over $X$. For any $\trianglerighteq \in \mathscr{P}$, define preference function $A : \mathscr{P} \to \mathscr{P}$ as

$$x' A(\succeq)\, x \ \text{ if } \ \mu\big(L_{\trianglerighteq}(x')\big) + \mu\big(L_{\succeq}(x')\big) \geq \mu\big(L_{\trianglerighteq}(x)\big) + \mu\big(L_{\succeq}(x)\big).$$

Note that, for any $\succeq \in \mathscr{P}$ and $x \in X$, $\mu\big(L_{\succeq}(x)\big) \cdot |X|$ determines the number of alternatives worse than $x$ with respect to $\succeq$ and, thus, the rank of the alternative $x$.[6] Therefore, as in Borda rule, the objective $A(\succeq)$ of the parent is determined by evaluating the sum of rankings assigned to each alternative according to some "core" preferences $\trianglerighteq$ of the parent and the preferences $\succeq$ of the descendant. The alternative for which the sum of the rankings is higher is the one eventually selected by the parent.

Each of the preference functions discussed in Examples 2–5 is anchored. Indeed, whenever $\mathcal{I} \in \mathscr{P}$, both conformism and contrarianism are anchored to $\trianglerighteq = \mathcal{I}$. The egocentric preference function is trivially anchored to its unique value $\trianglerighteq$. Similarly, the Borda rule preference function is also anchored to $\trianglerighteq$.

All the examples provided so far belong to a wider class of preference functions that we dub *Borda externality*, specified as in the example below.

**Example 6** (Borda externality)**.** Let $X$ be finite and $\mu$ be a probability measure over $X$ with the full support. For any $f : X \times [0,1] \to \mathbb{R}$, define the function $A : \mathscr{P} \to \mathscr{P}$ by

$$x' A(\succeq)\, x \ \text{ if } \ f\Big(x', \mu\big(L_{\succeq}(x')\big)\Big) \geq f\Big(x, \mu\big(L_{\succeq}(x)\big)\Big).$$

Therefore, the utility of the parent depends directly on the chosen alternative and the well-being of the descendant captured through the function $x \to \mu\big(L_{\succeq}(x)\big)$. Clearly, whenever $f(x, m) = \mu\big(L_{\trianglerighteq}(x)\big) + m$, for some $\trianglerighteq \in \mathscr{P}$, we obtain the Borda rule.

[6] In fact, $x \to \mu\big(L_{\succeq}(x)\big)$ is a utility representation of $\succeq$.

Generalisation of preference functions with Borda externalities to more abstract spaces is non-trivial. This is because the function $x \to \mu\big(L_{\trianglerighteq}(x)\big)$ is not well-behaved and, in some cases, may not be well-defined. As a result, we cannot guarantee that the function $A$, defined in Example 6, is well-defined itself. In Section A.1 of the Appendix, we remedy this by modifying our measurement of the lower contour sets.

It is straightforward to show that a preference $A$ represented with Borda externality is anchored whenever $f(x', m) \geq f(x, m)$ is equivalent to $f(x', m') \geq f(x, m')$, for any $x, x' \in X$ and $m, m' \in [0, 1]$. In fact, whenever the sets $X$ and $\mathscr{P}$ are rich enough, this condition is also necessary. See Proposition A.4 in the Appendix.

**Remark 1.** Regarding the preference function in Example 6, one could always consider an arbitrary "utility formula" $U : X \times \mathscr{P} \to \mathbb{R}$, where $x \to U(x, \succeq)$ is a utility representation of $\succeq$, and define the preference function $A : \mathscr{P} \to \mathscr{P}$ as

$$x' A(\succeq)\, x \ \text{ if } \ f\big(x', U(x', \succeq)\big) \geq f\big(x, U(x, \succeq)\big),$$

for some $f : X \times \mathbb{R} \to \mathbb{R}$, assuming that $f$ is chosen in a way that makes $A$ well-defined, i.e., we have $A(\succeq) \in \mathscr{P}$, for any $\succeq \in \mathscr{P}$.

However, in order for $A$ to be a function, it is critical to specify the *exact* mapping $(x, \succeq) \to U(x, \succeq)$, since for any distinct $U$, $U'$, the functions $x \to f\big(x, U(x, \succeq)\big)$ and $x \to f\big(x, U'(x, \succeq)\big)$ would induce different preferences—even if $x \to U(x, \succeq)$ and $x \to U'(x, \succeq)$ represented the same relation $\succeq$. Thus, when defining $A$, the choice of the function $U$ is equally important as the choice of the function $f$. In our examples, we propose $U(x, \succeq) := \mu\big(L_{\succeq}(y)\big)$, which is well-defined under certain conditions.[7]

To make our point concrete, suppose that $X = \{0, 1\}$ and $f(x, m) = x + m$. Let $0 \succ 1$ and consider two functions $U(x, \succeq) = -x/2$ and $U'(x, \succeq) = -2x$. Clearly, both functions represent $\succeq$. However, since $f\big(1, U(1, \succeq)\big) = 1/2 > 0 = f\big(0, U(0, \succeq)\big)$ and $f\big(1, U'(1, \succeq)\big) = -1 < 0 = f\big(0, U'(0, \succeq)\big)$, the two functions induce different rankings and, thus, correspond to different preference functions $A$, $A'$.

So far, all of our examples were a special case of Borda externality. Below, we present preference functions that are not representable in that way.

[7] Other "utility formulas" are possible. For example, if $X = \mathbb{R}^{\ell}_{+}$ and $\mathscr{P}$ consists of upper semi-continuous and locally non-satiated preferences, one could define $U(x, \succeq) := \min\big\{\sum_{i=1}^{\ell} y_i : y \succeq x\big\}$. However, any cardinalisation of preferences necessarily affects the values of the function $A$.

**Example 7.** Suppose that $X$ is finite and $\mu$ is a probability measure over $X$. For any $\trianglerighteq \in \mathscr{P}$, define the function $A : \mathscr{P} \to \mathscr{P}$ as

$$xA(\succeq)y \;\text{ if }\; \mu\big(L_{\trianglerighteq}(x) \cap L_{\succeq}(x)\big) \geq \mu\big(L_{\trianglerighteq}(y) \cap L_{\succeq}(y)\big).$$

Therefore, the parent prefers alternatives over which they agree the most with the descendant—in terms of cardinality of the intersection of lower contour sets. Alternatively, one could define $xA'(\succeq)y$ if $\mu\big(L_{\trianglerighteq}(x) \cup L_{\succeq}(x)\big) \geq \mu\big(L_{\trianglerighteq}(y) \cup L_{\succeq}(y)\big)$. It is easy to show that both functions are anchored to $\trianglerighteq$.

# 3 Comparative altruism

When can we say that one parent is more altruistic than another? In this section we formalise and characterise this notion. First, we need to introduce some additional notation. Given preferences $\succeq$ and a set $B \subseteq X$, denote the choice set by

$$\Phi_{\succeq}(B) \;:=\; \Big\{x \in B : x \succeq y, \text{ for all } y \in B\Big\}.$$

Whenever $\succeq$ is upper semi-continuous and $B$ is non-empty and compact, the set $\Phi_{\succeq}(B)$ is non-empty. For this reason, throughout the remainder of this paper, we assume that $\mathscr{P}$ consists of upper semi-continuous preferences.

## 3.1 "More altruistic" behaviour

We start by defining the notion of a "more altruistic" ranking over preference functions.

**Definition 3** (More altruistic)**.** A preference function $A'$ is *more altruistic* than $A$ if, for any $\succeq \in \mathscr{P}$, any non-empty and compact set $B \subseteq X$, and any $x \in \Phi_{A(\succeq)}(B)$ and $x' \in \Phi_{A'(\succeq)}(B)$, there is $y \in \Phi_{A(\succeq)}(B)$ and $y' \in \Phi_{A'(\succeq)}(B)$, such that $x' \succeq y$ and $y' \succeq x$.

Intuitively speaking, one parent is more altruistic than another if the descendant prefers choices of the former to choices of the latter. Specifically, our definition requires that, for *any* preferences $\succeq \in \mathscr{P}$ of the descendant and any compact menu $B$, the choice set $\Phi_{A'(\succeq)}(B)$ of the more altruistic parent dominates the choice set $\Phi_{A(\succeq)}(B)$ of the other with respect to the *weak set order* induced by the preference relation $\succeq$. That is, for any choice of the less altruistic parent, there is a choice of the more altruistic one that is preferable to the descendant, and vice versa—for any choice of the latter, there is a choice of the former that would make the descendant worse off.

Below, we characterise preference functions ordered in the above sense.

**Theorem 1.** *For any functions $A, A' : \mathscr{P} \to \mathscr{P}$, these statements are equivalent.*

*(i) The function $A'$ is more altruistic than $A$.*

*(ii) For any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succ x$,*

$$x' \succeq_A (\succ_A)\, x \quad \textit{implies} \quad x' \succeq_{A'} (\succ_{A'})\, x,$$

*where $\succeq_A := A(\succeq)$ and $\succeq_{A'} := A'(\succeq)$.*

*(iii) For any $\succeq \in \mathscr{P}$ and $x \in X$, we have $U_{A(\succeq)}(x) \cap U^\circ_{\succeq}(x) \subseteq U_{A'(\succeq)}(x) \cap U^\circ_{\succeq}(x)$, and $U^\circ_{A(\succeq)}(x) \cap U^\circ_{\succeq}(x) \subseteq U^\circ_{A'(\succeq)}(x) \cap U^\circ_{\succeq}(x)$.*

*(iv) For any $\succeq \in \mathscr{P}$ and $x \in X$, we have $L_{A(\succeq)}(x) \cap L^\circ_{\succeq}(x) \subseteq L_{A'(\succeq)}(x) \cap L^\circ_{\succeq}(x)$, and $L^\circ_{A(\succeq)}(x) \cap L^\circ_{\succeq}(x) \subseteq L^\circ_{A'(\succeq)}(x) \cap L^\circ_{\succeq}(x)$.*

*(v) For any $\succeq \in \mathscr{P}$, any non-empty and compact set $B \subseteq X$, and any $x \in \Phi_{A(\succeq)}(B)$, $x' \in \Phi_{A'(\succeq)}(B)$, if $x \succ x'$ then $x' \in \Phi_{A(\succeq)}(B)$ and $x \in \Phi_{A'(\succeq)}(B)$.*

*Proof.* We prove implication (i) $\Rightarrow$ (ii) by contradiction. Suppose that $A'$ is more altruistic than $A$, but there is some $x, x' \in X$ such that $x' \succ x$ and either (a) $x' \succeq_A x$ and $x \succ_{A'} x'$, or (b) $x' \succ_A x$ and $x \succeq_{A'} x'$. Let $B = \{x, x'\}$. In case (a), we have $x' \in \Phi_{A(\succeq)}(B)$ and $\Phi_{A'(\succeq)}(B) = \{x\}$. Therefore, there is no $y' \in \Phi_{A'(\succeq)}(B)$ such that $y' \succeq x'$. In case (b), we have $\Phi_{A(\succeq)}(B) = \{x'\}$ and $x \in \Phi_{A'(\succeq)}(B)$. Thus, there is no $y \in \Phi_{A(\succeq)}(B)$ such that $x \succeq y$. Either way, this contradicts that $A'$ is more altruistic than $A$.

To show that (ii) $\Rightarrow$ (iii), take any $\succeq \in \mathscr{P}$ and $x \in X$. If the set $U_{A(\succeq)}(x) \cap U^\circ_{\succeq}(x)$ is empty, we are done. Otherwise, take any $x' \in U_{A(\succeq)}(x) \cap U^\circ_{\succeq}(x)$. By construction, we have $x' \succ x$ and $x' A(\succeq)\, x$, which implies $x' A'(\succeq)\, x$ and, thus $x' \in U_{A'(\succeq)}(x) \cap U^\circ_{\succeq}(x)$. We prove the second part of the claim analogously.

To show that (iii) implies (iv), take any $\succeq \in \mathscr{P}$ and $x \in X$. If the set $L_{A(\succeq)}(x) \cap L^\circ_{\succeq}(x)$ is empty, we are done. If not, take any $x' \in L_{A(\succeq)}(x) \cap L^\circ_{\succeq}(x)$. Note that this is equivalent to $x \in U_{A(\succeq)}(x') \cap U^\circ_{\succeq}(x')$. By statement (iii), we have $x \in U_{A'(\succeq)}(x') \cap U^\circ_{\succeq}(x')$, which implies $x' \in L_{A'(\succeq)}(x) \cap L^\circ_{\succeq}(x)$. We prove the second part analogously.

Next, we show that (iv) implies (v). Take any $\succeq \in \mathscr{P}$, any non-empty and compact set $B \subseteq X$, and any $x \in \Phi_{A(\succeq)}(B)$, $x' \in \Phi_{A'(\succeq)}(B)$ such that $x \succ x'$. Since $x \in \Phi_{A(\succeq)}(B)$,

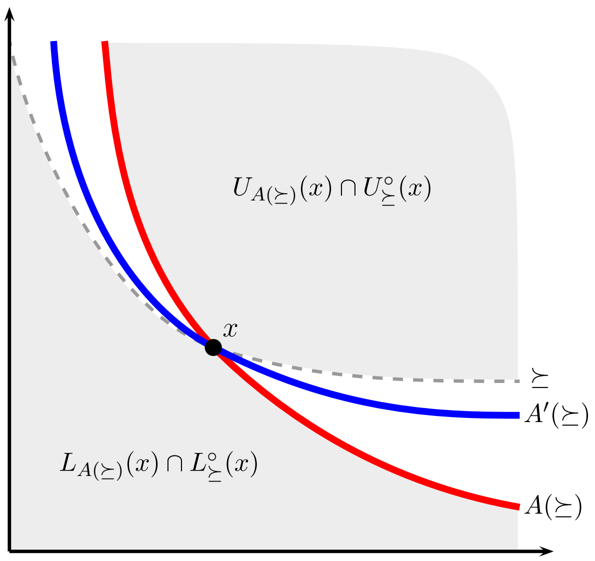


Figure 1: Preference function $A'$ is more altruistic than $A$.

we have $x' \in L_{A(\succeq)}(x) \cap L^{\circ}_{\succeq}(x)$, which implies $x' \in L_{A'(\succeq)}(x)$ and, thus, $x \in \Phi_{A'(\succeq)}(B)$. Similarly, whenever $x' \notin \Phi_{A(\succeq)}(B)$, it must be that $x' \in L^{\circ}_{A(\succeq)}(x) \cap L^{\circ}_{\succeq}(x)$, which implies $x' \in L^{\circ}_{A'(\succeq)}(x)$. However, this contradicts that $x' \in \Phi_{A'(\succeq)}(B)$.

Finally, we prove that (v) implies (i). Take any $\succeq \in \mathscr{P}$, $B \subseteq X$, and $x, x'$ as in statement (v). If $x' \succeq x$, we are done. If $x \succ x'$, statement (v) implies that $x' \in \Phi_{A(\succeq)}(B)$ and $x \in \Phi_{A'(\succeq)}(B)$, hence, the function $A'$ is more altruistic than $A$. $\square$

Theorem 1 characterises the "more altruistic" ordering in two ways. Statement (ii) stipulates that two preference functions $A$, $A'$ are ordered in the "more altruistic" sense if, and only if, their values $A(\succeq)$, $A'(\succeq)$ are ordered according to the single-crossing condition à la Spence-Mirrlees, induced by the strict part of the descendant's preference relation $\succeq$.[8] The characterisation in (iii) and (iv) relates the notion of altruism to the intersection of contour sets. Specifically, the intersection of the descendant's (lower-) upper-contour set with the (lower-) upper-contour set of the less altruistic parent is always a subset of the corresponding intersection between the (lower-) upper-contour set of the descendant and the more altruistic parent. Therefore, preferences of the more altruistic parent are always closer to the descendant's, in the sense of overlapping contour sets. See Figure 1 for a graphical interpretation of this condition.

Finally, statement (v) in Theorem 1 posits that the "more altruistic" ordering is equiv-

[8] See also the single-crossing condition in Milgrom and Shannon (1994). However, note that in our case $\succeq$ is not a partial order, but a complete pre-order and $X$ need not be a lattice.

alent to an alternative relation between the choice sets induced by $A$ and $A'$. Whenever an element of $\Phi_{A(\succeq)}(B)$ is strictly preferable to an element of $\Phi_{A'(\succeq)}(B)$, the former must belong to $\Phi_{A'(\succeq)}(B)$ and the latter must be in $\Phi_{A(\succeq)}(B)$.[9] This seemingly stronger condition is equivalent to being more altruistic in our setting.

Theorem 1 characterises comparative statics for general preference functions. However, our characterisation becomes stronger once we focus on anchored preferences.

**Proposition 2.** *Let $\mathscr{P}$ be indifference-rich and the functions $A, A' : \mathscr{P} \to \mathscr{P}$ be anchored to some $\trianglerighteq, \trianglerighteq' \in \mathscr{P}$, respectively. These statements are equivalent.*

*(i) The function $A'$ is more altruistic than $A$ and $\trianglerighteq = \trianglerighteq'$.*

*(ii) For any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succeq x$,*

$$x' \succeq_A (\succ_A)\ x \quad \textit{implies} \quad x' \succeq_{A'} (\succ_{A'})\ x,$$

*where $\succeq_A := A(\succeq)$ and $\succeq_{A'} := A'(\succeq)$.*

*(iii) For any $\succeq \in \mathscr{P}$ and $x \in X$, we have $U_{A(\succeq)}(x) \cap U_{\succeq}(x) \subseteq U_{A'(\succeq)}(x) \cap U_{\succeq}(x)$, and $U^{\circ}_{A(\succeq)}(x) \cap U_{\succeq}(x) \subseteq U^{\circ}_{A'(\succeq)}(x) \cap U_{\succeq}(x)$.*

*(iv) For any $\succeq \in \mathscr{P}$ and $x \in X$, we have $L_{A(\succeq)}(x) \cap L_{\succeq}(x) \subseteq L_{A'(\succeq)}(x) \cap L_{\succeq}(x)$, and $L^{\circ}_{A(\succeq)}(x) \cap L_{\succeq}(x) \subseteq L^{\circ}_{A'(\succeq)}(x) \cap L_{\succeq}(x)$.*

*(v) For any $\succeq \in \mathscr{P}$, any non-empty and compact set $B \subseteq X$, and any $x \in \Phi_{A(\succeq)}(B)$, $x' \in \Phi_{A'(\succeq)}(B)$, if $x \succeq x'$ then $x' \in \Phi_{A(\succeq)}(B)$ and $x \in \Phi_{A'(\succeq)}(B)$.*

*Proof.* Implication (i) $\Rightarrow$ (ii) follows from Theorem 1 and the fact that $x' A(\succeq) x$ if, and only if $x' A'(\succeq) x$ if, and only if, $x' \trianglerighteq x$, for any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x \sim x'$.

Implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) can be shown as in Theorem 1. Finally, by Theorem 1, we know that (v) implies that $A'$ is more altruistic than $A$. We need to show that $\trianglerighteq = \trianglerighteq'$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x \sim x'$. Define $B = \{x, x'\}$. By (v), it must be that $\Phi_{A(\succeq)}(B) = \Phi_{A'(\succeq)}(B)$. Hence, we have $x \trianglerighteq x'$ if, and only if, $x \trianglerighteq' x'$. As a result, by Proposition 1, it must be that $\trianglerighteq = \trianglerighteq'$. □

The above characterisation is almost equivalent to the one presented in Theorem 1. However, in the latter result, each condition has to be verified for any two alternatives that are *weakly* ranked by the descendant's preference relation $\succeq$, rather than *strictly*, as in Theorem 1. Although the change seems to be minor, it will have important implications regarding properties of the "more altruistic" relation.

[9] This property is analogous to the *strong set order*. See, e.g., Topkis (1979).

## 3.2 Properties of the "more altruistic" relation

In this section, we discuss properties of the "more altruistic" relation.

**Proposition 3.** *The "more altruistic" relation is a preorder (i.e., reflexive and transitive).*

*Proof.* Reflexivity of the relation is immediate. To prove its transitivity, take any preference functions $A, A', A''$ such that $A'$ is more altruistic than $A$, and $A''$ is more altruistic than $A'$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$, such that $x' \succ x$. By Theorem 1, $x' \succeq_A (\succ_A)\, x$ implies $x' \succeq_{A'} (\succ_{A'})\, x$, which implies $x' \succeq_{A''} (\succ_{A''})\, x$, where we denote $\succeq_A := A(\succeq)$, $\succeq_{A'} := A'(\succeq)$, and $\succeq_{A''} := A''(\succeq)$. Therefore, $A''$ is more altruistic than $A$. $\square$

In general, the "more altruistic" relation is *not* a partial order, since it need not be antisymmetric. For example, suppose that $X = \{x, x'\}$ and $\mathscr{P} = \{\succeq^1, \succeq^2, \succeq^3\}$, where $x' \succ^1 x$, $x \succ^2 x'$, $x \sim^3 x'$. Take preference functions $A, A'$ such that $A(\succeq^i) = A'(\succeq^i) = \succeq^i$, for $i = 1, 2$, and $A(\succeq^3) = \succeq^1$, $A'(\succeq^3) = \succeq^2$. Clearly $A \neq A'$, however, both $A'$ is more altruistic than $A$, and $A$ is more altruistic than $A'$. This is because, for $\succeq^3$, the condition in Theorem 1(ii) is trivially satisfied when $x \sim^3 x'$.

As follows from the example above, the "more altruistic" relation fails to be antisymmetric precisely because the single-crossing condition stated in Theorem 1(ii) holds trivially for all those alternatives that the descendant finds indifferent. However, there are settings in which antisymmetry holds. Consider the following result.

**Proposition 4.** *Suppose that $X = \mathbb{R}^\ell_+$ and $\mathscr{P}$ consists of continuous preferences such that $x' \gg x$ implies $x' \succ x$, for all $\succeq \in \mathscr{P}$.*[10] *Then, any set of preference functions defined over $\mathscr{P}$ is partially ordered by the "more altruistic" relation.*[11]

*Proof.* It suffices to show that the relation is antisymmetric. Suppose that the function $A'$ is more altruistic than $A$ and vice versa. By Theorem 1(ii), for any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succ x$, we have $x' A(\succeq)\, x$ if, and only if $x' A'(\succeq)\, x$.

We need to show that the same holds if $x \sim x'$. Towards contradiction, suppose that $x' A(\succeq)\, x$ and $x \succ_{A'} x'$, where $\succeq_{A'} = A'(\succeq)$. Take any sequence $y^k$ converging to $x'$, such that $y^k \gg x'$, for all $k$. Since $\succeq$ and $A(\succeq)$ are consistent with $\gg$, we have $y^k \succ x' \sim x$ and $y^k \succ_A x' \succeq_A x$, for all $k$, where $\succeq_A = A(\succeq)$. Since $A'$ is more altruistic than $A$,

[10] We have $x' \gg x$ whenever $x'_i > x_i$, for all $i = 1, \dots, \ell$.

[11] In fact, this claim remains true whenever $\mathscr{P}$ consists of locally non-satiated preferences such that $L^\circ_{\succeq}(x) \cap L^\circ_{\succeq'}(x) \neq \emptyset$, for all $x \in X$ and $\succeq, \succeq' \in \mathscr{P}$.

we have $y^k \succ'_A x$, for all $k$. By continuity of $A'(\succeq)$, this implies $x' A'(\succeq)\, x$, yielding a contradiction. Therefore, we have $x' A(\succeq)\, x$ if, and only if $x' A'(\succeq)\, x$. □

Another example for which the "more altruistic" relation is antisymmetric is a space of preference functions anchored to the same relation $\trianglerighteq \in \mathscr{P}$.

**Proposition 5.** *Suppose that $\mathscr{P}$ is indifference-rich. For any $\trianglerighteq \in \mathscr{P}$, the set of preference functions anchored to $\trianglerighteq$ is partially ordered by the "more altruistic" relation.*

*Proof.* We only need to show that the "more altruistic than" relation is antisymmetric. Take any $A, A'$ such that $A$ dominates $A'$ and vice versa. We need to show that $A = A'$. Towards contradiction, suppose that $x' \succeq x$, $x' \succeq_A x$, and $x \succ_{A'} x'$, for some $\succeq \in \mathscr{P}$ and $x, x' \in X$. Since $A'$ is more altruistic than $A$, by Proposition 2, it must be that $x' \succeq x$ and $x' \succeq_A x$ implies $x' \succeq_{A'} x$, which yields a contradiction. □

## 3.3 Examples

In this subsection we provide examples of classes of functions that are ordered in the "more altruistic" sense. First, we discuss bounds of the space of preference functions.

**Proposition 6.** *For any $\mathscr{P}$, the conformistic preference function $A(\succeq) := \succeq$ is more altruistic than any other preference function.*

*Proof.* Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succ x$. In particular, we have $x' \succ_A x$, where $\succeq_A := A(\succeq)$. By Theorem 1(ii), this suffices for $A$ to dominate any function. □

Since the "more altruistic" relation is not antisymmetric, conformism need not be the unique upper bound. However, following Proposition 4, it is the unique upper bound of the space of strictly increasing preferences over $\mathbb{R}^\ell_+$. Similarly, following Proposition 5, it is the unique upper bound of the set of all functions anchored to $\mathcal{I}$.

Our next result determines the lower bound for the space of preference functions.

**Proposition 7.** *Let $\mathscr{P}$ consist of all preferences over $X$. Any preference function is more altruistic than the contrarian (see Example 3).*

*Proof.* Denote the contrarian function by $A$, and let $A'$ be any preference function over $\mathscr{P}$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succ x$. Therefore, we have $x \succ_A x'$, where $\succeq_A := A(\succeq)$. By Theorem 1(ii), $A$ is always dominated by $A'$. □

Proposition 7 requires some comment. Recall that, unlike conformism, contrarianism need not be well-defined over an arbitrary set $\mathscr{P}$. Indeed, it must be that, for any $\succeq \in \mathscr{P}$, the set $\mathscr{P}$ admits a preference relation $\succeq'$, given by: $x' \succeq' x$ only if $x \succeq x$. As a result, we cannot guarantee that contrarianism is the unique lower bound over the space of strictly increasing preferences like in Proposition 4. This is because, for any $\succeq \in \mathscr{P}$, its counterpart $\succeq'$, defined as above, is not strictly increasing.

Next, we turn to egocentric preference functions, and show that none is more altruistic than another. Hence, the set of paternalistic functions is completely unordered.

**Proposition 8.** *Suppose that if $x' \succeq x$, for some $\succeq \in \mathscr{P}$, then $x' \succ' x$, for some $\succeq' \in \mathscr{P}$. Then, two egocentric preference functions are ordered in the more altruistic sense if, and only if, they are equal.*

*Proof.* Implication ($\Leftarrow$) follows from the "more altruistic than" relation being reflexive. To show the converse, take any two egocentric preference functions $A(\succeq) := \trianglerighteq$ and $A(\succeq) := \trianglerighteq'$, for some $\trianglerighteq, \trianglerighteq' \in \mathscr{P}$. Towards contradiction, suppose that $\trianglerighteq \neq \trianglerighteq'$. Hence, there is $x, x' \in X$ such that $x' \vartriangleright x$ and $x \trianglerighteq' x'$. Let $\succeq = \trianglerighteq$. Then, we have $x' \succ x$ and $x' \succ_A x$, but $xA'(\succeq)x'$, where $\succeq_A := A(\succeq) = \trianglerighteq$ and $\succeq_{A'} := A'(\succeq) = \trianglerighteq'$. Hence, $A'$ does not dominate $A$. Similarly, since $x \trianglerighteq' x'$, by assumption, there is $\succeq \in \mathscr{P}$ such that $x \succ x'$. However, given that $xA'(\succeq)x'$ and $x' \succ_A x$, $A$ does not dominate $A'$. $\square$

The reason why no paternalistic/egocentric preference function is more altruistic than another, follows precisely from them being constant and not adjusting objectives of the parent to the preferences of the descendant. As a result, one can always find a descendant who prefers one parent to the other, and vice versa.

The assumption imposed in Proposition 8 guarantees a rich enough domain. Without it, two paternalistic preference functions could be ordered, as shown below.

**Example 8.** Let $X = \{x, y\}$ and $\mathscr{P} = \{\succeq^1, \succeq^2\}$, where $x \succ^1 y$ and $x \sim^2 y$. In particular, even though we have $y \succeq x$, for some $\succeq \in \mathscr{P}$, there is no $\succeq'$ satisfying $x \succ' y$. Hence, the assumption from in Proposition 8 does not hold. As a result, one can define two paternalistic preference functions $A, A' : \mathscr{P} \to \mathscr{P}$, where $A(\succeq) := \succeq^2$ and $A'(\succeq) := \succeq^1$, such that $A'$ is strictly more altruistic than $A$.

Finally, we turn to preference functions that admit Borda externality representation. For any two functions $A, A'$ within this class, more altruistic behaviour is equivalent to

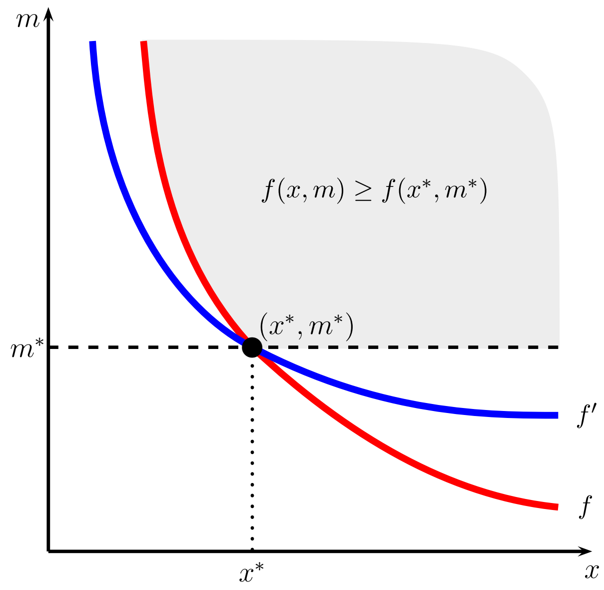


Figure 2: Aggregator $f'$ is more altruistic than $f$.

the aggregate functions $f, f'$, respectively, being ordered according to the single-crossing condition à la Spence-Mirrlees. See Figure 2.

**Proposition 9.** *For any two preference functions $A, A' : \mathscr{P} \to \mathscr{P}$ defined as in Examples 6, for some aggregators $f, f' : X \times [0,1] \to \mathbb{R}$, respectively, the function $A'$ is more altruistic than $A$ if, for any $x, x' \in X$ and $m' > m$,*

$$f\big(x', m'\big) \geq (>)\ f\big(x, m\big) \quad \textit{implies} \quad f'\big(x', m'\big) \geq (>)\ f'\big(x, m\big).$$

*Proof.* Take any $\succeq\ \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succ x$. Denote $m = \mu\big(L_{\succeq}(x)\big)$, $m' = \mu\big(L_{\succeq}(x')\big)$. Since $z \to \mu\big(L_{\succeq}(z)\big)$ is a utility function, we have $m' > m$. Suppose that $x' \succeq_A (\succ_A)\ x$, where $\succeq_A := A(\succeq)$. As $f$ represents $A$ in the sense of Borda externality, we have $f(x', m') \geq (>)\ f(x, m)$. By assumption, this implies $f'(x', m') \geq (>)\ f'(x, m)$ and, thus, $x' \succeq_{A'} (\succ_{A'})\ x$, where $\succeq_{A'} := A'(\succeq)$. Hence, $A'$ is more altruistic than $A$. □

Whenever the sets $X$ and $\mathscr{P}$ are rich enough, the condition stated in Proposition 9 is also necessary for two preference functions represented with Borda externality to be ordered in the more altruistic sense. See Proposition A.5 in the Appendix.

The next application follows directly from Proposition 9.

**Example 5' (Weighted Borda rule).** Suppose that $X$ is finite and let $\mu$ be a probability measure over $X$. Consider a preference function $A^{\alpha} : \mathscr{P} \to \mathscr{P}$ defined by:

$$x' A^{\alpha}(\succeq)\, x \ \text{ if } \ \mu\big(L_{\trianglerighteq}(x')\big) + \alpha\mu\big(L_{\succeq}(x')\big) \geq \mu\big(L_{\trianglerighteq}(x)\big) + \alpha\mu\big(L_{\succeq}(x)\big),$$

for some $\trianglerighteq \in \mathscr{P}$ and $\alpha \in \mathbb{R}$. Given Proposition 9, one can easily show that the preference function $A^{\alpha'}$ is more altruistic than $A^{\alpha}$ whenever $\alpha' \geq \alpha$. Following Proposition A.5 in the Appendix, this condition is also necessary whenever the domain $\mathscr{P}$ is rich enough.

# 4 Absolute altruism (and antagonism)

In this section we define and explore a notion of absolute altruism. Throughout this section we focus on preference functions that are anchored to some $\trianglerighteq \in \mathscr{P}$. We say that a preference function is *altruistic*, if it is more altruistic than an egocentric parent endowed with the preference $\trianglerighteq$. That is, an altruist is someone who behaves more favourably towards others relatively to their behaviour in isolation.

**Definition 4** (Altruism/Antagonism)**.** The preference function $A$ is *altruistic* (*antagonistic*) if it is anchored to some $\trianglerighteq \in \mathscr{P}$ and is more altruistic (less altruistic) than the egocentric/paternalistic preference function given by $A'(\succeq) := \trianglerighteq$, for all $\succeq \in \mathscr{P}$.

By definition, a parent is altruistic whenever their actions are more favourable to the descendant than choices of an egocentric endowed with the same core preferences.

## 4.1 Characterising absolute altruism

We devote this subsection to various characterisation of absolute altruism and antagonism, as well as some of its more important properties.

**Theorem 2.** *For any function $A : \mathscr{P} \to \mathscr{P}$, the following statements are true.*

*(i) The function $A$ is altruistic if, and only if, there is some $\trianglerighteq \in \mathscr{P}$ such that, for any $\succeq \in \mathscr{P}$ and $x, x' \in X$, $x' \succeq x$ and $x' \trianglerighteq (\triangleright)\, x$ implies $x' \succeq_A (\succ_A)\, x$.*

*(ii) The function $A$ is antagonistic if, and only if, there is some $\trianglerighteq \in \mathscr{P}$ such that, for any $\succeq \in \mathscr{P}$ and $x, x' \in X$, $x' \succeq x$ and $x' \succeq_A (\succ_A)\, x$ implies $x' \trianglerighteq (\triangleright)\, x$.*

*Proof.* We only prove statement (i), as (ii) can be shown analogously. To prove implication ($\Rightarrow$), suppose that $A$ is anchored to some $\trianglerighteq \in \mathscr{P}$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$. If $x \sim x'$, then $x' \trianglerighteq (\triangleright)\, x$ if, and only if, $x' \succeq_A (\succ_A)\, x$, since $A$ is anchored to $\trianglerighteq$. If $x' \succ x$, then $x' \trianglerighteq (\triangleright)\, x$ implies $x' \succeq_A (\succ_A)\, x$, given that $A$ is altruistic.

To show the converse, first, we prove that $A$ is anchored to $\trianglerighteq$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x \sim x'$. By assumption, $x' \trianglerighteq (\triangleright)\, x$ implies $x' \succeq_A (\succ_A)\, x$. Suppose that $x' \succeq_A x$ but $x \rhd x'$. However, then $x \sim x'$ and $x \rhd x'$ would require that $x \succ_A x$, yielding a contradiction. Similarly, $x' \succ_A x$ must imply $x \rhd x'$. Finally, by Theorem 1(ii), $A$ is more altruistic than the paternalistic function taking the constant value of $\trianglerighteq$. $\square$

This theorem highlights what it means to be altruistic and antagonistic. An altruist is a parent who always follows preferences of the descendant, as long as they do *not* contradict their own core preference $\trianglerighteq$. In its essence, altruism is equivalent to the objective $A(\succeq)$ of the parent satisfying the *Pareto principle* with respect to their core preferences and the preferences of the descendant. An antagonist, on the other hand, never follows preferences of the descendant, *unless* they agree with their own. Equivalently, they go against their core preferences only if it makes the descendant worse off.

The characterisation in Theorem 2 highlights the connection between the values of function $A$ and the preferences $\trianglerighteq$ to which it is anchored. Below, we push our characterisation further to understand the connection between values of the function $A$ evaluated at different elements in $\mathscr{P}$. Consider the following proposition.

**Proposition 10.** *Let $\mathcal{I} \in \mathscr{P}$. For any $A : \mathscr{P} \to \mathscr{P}$, these statements are true.*

*(i) The function $A$ is* altruistic *if, and only if, whenever $x' \succeq x$ and $x \succ_A x'$, for some $x, x' \in X$ and $\succeq \in \mathscr{P}$, then $x \succeq' x'$ implies $x \succ'_A x'$, for all $\succeq' \in \mathscr{P}$.*

*(ii) The function $A$ is* antagonistic *if, and only if, whenever $x' \succeq x$ and $x' \succeq_A x$, for some $x, x' \in X$ and $\succeq \in \mathscr{P}$, then $x \succeq' x'$ implies $x' \succeq'_A x$, for all $\succeq' \in \mathscr{P}$.*

*Above, we denote $\succeq_A := A(\succeq)$ and $\succeq'_A := A(\succeq')$.*

*Proof.* First, we prove statement (i). To show implication ($\Rightarrow$), suppose that $A$ is anchored to some relation $\trianglerighteq \in \mathscr{P}$. Whenever $x' \succeq x$ and $x \succ_A x'$, for some $x, x' \in X$ and $\succeq \in \mathscr{P}$, then it must be that $x \rhd x'$. Otherwise, it would have to be $x' A(\succeq)\, x$, since $A$ is altruistic (by Theorem 2). Thus, $x \succeq' x'$ implies $x \succ'_A x'$, for any $\succeq' \in \mathscr{P}$.

To prove the converse, it suffices to show that $A$ obeys condition (i) in Theorem 2 for $\trianglerighteq := A(\mathcal{I})$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succeq x$. Towards contradiction, suppose that $x' \trianglerighteq x$ and $x \succ_A x'$. By assumption, $x' \succeq x$ and $x \succ_A x'$ requires that $x \succeq' x'$ implies $x \succ'_A x'$, for any $\succeq' \in \mathscr{P}$. However, we have $x\,\mathcal{I}\,x'$ and $x' A(\mathcal{I})\, x$. Alternatively, suppose that $x' \rhd x$. In particular, this implies $x\mathcal{I}x'$ and not $xA(\mathcal{I})x'$. Therefore, by assumption, $x' \succeq x$ implies $x' \succ_A x$, for any $\succeq \in \mathscr{P}$, where $\succeq_A := A(\succeq)$.

Next, we prove statement (ii). To show implication ($\Rightarrow$), suppose that $A$ is anchored to some relation $\trianglerighteq \in \mathscr{P}$. By condition (ii) in Theorem 2, $x' \succeq x$ and $x' \succeq_A x$ implies $x' \trianglerighteq x$, for any $\succeq \in \mathscr{P}$ and $x, x' \in X$. Therefore, whenever $x \succeq' x'$, for some $\succeq' \in \mathscr{P}$, then it must be that $x' \succeq'_A x$. Otherwise, it would imply $x \vartriangleright x'$, yielding a contradiction.

To prove the converse, it suffices to show that $A$ satisfies condition (ii) in Theorem 2 for $\trianglerighteq := A(\mathcal{I})$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succeq x$. If $x' \succeq_A x$, then $x\mathcal{I}x'$ must imply $x' A(\mathcal{I})\, x$, which is equivalent to $x' \trianglerighteq x$. Alternatively, if $x' \succ_A x$ and $xA(\mathcal{I})x'$, then $x\mathcal{I}x'$ would require that $x' \succeq x$ implies $x \succeq_A x'$, yielding a contradiction. Therefore, it must be that $x' \succ_A x$ implies $x' \vartriangleright x$, which completes the proof. $\square$

Proposition 10 provides a deeper understanding of altruism and antagonism. What does it mean to be an altruist? It means that, whenever you go against preferences of a descendant and choose some $x'$ over $x$, then you would never go against a descendant who would agree with this decision, i.e., prefer $x'$ over $x$. Alternatively, if an antagonist agrees with a descendant and chooses some $x'$ over $x$, then they would always (weakly) contradict any descendant who would prefer $x$ over $x'$.

## 4.2 Absolute altruism and comparative statics

When considering altruistic preferences, "more altruistic" is equivalent to the values of the function being farther away from the core preferences of the parent. Conversely, for antagonistic preferences, "more altruistic" is equivalent to the values of the function being closer to the preferences in isolation, as we show in the theorem below.

**Theorem 3.** *Let $A, A' : \mathscr{P} \to \mathscr{P}$ be preference functions anchored to the same $\trianglerighteq \in \mathscr{P}$. If $A$ and $A'$ are altruistic (antagonistic), the following statements are equivalent.*

*(i) The function $A'$ is more altruistic (less altruistic) than $A$.*

*(ii) For any $\succeq \in \mathscr{P}$ and $x, x' \in X$, $x' \trianglerighteq x$ and $x' \succeq_{A'} x$ implies $x' \succeq_A x$; while $x' \vartriangleright x$ and $x' \succ_{A'} x$ implies $x' \succ_A x$, where $\succeq_A := A(\succeq)$ and $\succeq_{A'} := A'(\succeq)$.*

*Proof.* First we show the result outside of the parentheses. To prove that (i) implies (ii), suppose that $A, A'$ are altruistic functions anchored to some $\trianglerighteq$. Take any $x, x' \in X$ such that $x' \trianglerighteq x$ and $x' \succeq_{A'} x$. Whenever $x' \succeq x$, then it must be that $x' \succeq_A x$, since $A$ is altruistic (see Theorem 2). Alternatively, suppose that $x \succ x'$. Towards contradiction, let $x \succ_A x'$. However, since $A'$ is more altruistic than $A$, $x \succ x'$ and $x \succ_A x'$ would imply $x \succ_{A'} x'$, yielding a contradiction. Analogously, $x' \rhd x$ and $x' \succ_{A'} x$ implies $x' \succ_A x$.

To show the converse, take any $\succeq \in \mathscr{P}$ and $x, x'$ such that $x' \succ x$. First, we show that $x'A(\succeq)x$ implies $x'A'(\succeq)x$. Whenever $x' \unrhd x$, this holds trivially, since $A'$ is altruistic. Conversely, suppose that $x \rhd x'$. If $x \succ_{A'} x'$ then, by assumption, it must be that $x \succ_A x'$, yielding a contradiction. Next, we show that $x' \succ_A x$ implies $x' \succ_{A'} x$. Whenever $x' \rhd x$, this holds trivially, since $A'$ is altruistic. If $x \unrhd x'$, then $x \succeq_{A'} x'$ would have to imply $x \succeq_A x'$, yielding a contradiction. Hence, $A'$ is more altruistic than $A$.

We proceed with the proof of the statement within the parentheses. To show implication ($\Rightarrow$), suppose that $A, A'$ are antagonistic functions anchored to some $\unrhd$. Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \unrhd x$ and $x'A'(\succeq)x$. Suppose that $x \succeq x'$. Since $A'$ is antagonistic, $x \succ_A x'$ would imply $x \rhd x'$, yielding a contradiction. Conversely, let $x' \succ x$. Since $A$ is more altruistic than $A'$, $x'A'(\succeq)x$ must imply $x'A(\succeq)x$. Analogously, we show that $x' \rhd x$ and $x' \succ_{A'} x$ implies $x' \succ_A x$. This proves our claim.

To prove the converse, take any $\succeq \in \mathscr{P}$ and $x, x'$ such that $x' \succ x$. First, we show that $x'A'(\succeq)x$ implies $x'A(\succeq)x$. Since $A'$ is antagonistic, $x' \succ x$ and $x'A'(\succeq)x$ implies $x' \unrhd x$. Therefore, by assumption, it must be that $x'A(\succeq)x$. We show that $x' \succ_{A'} x$ implies $x' \succ_A x$ analogously. Hence, $A$ is more altruistic than $A'$. $\square$

The result above allows us to interpret more altruistic behaviour in terms of how costly it is to depart from one's core preferences. The more altruistic a parent is, the easier (less costly) it is for them to agree with preferences of the descendant. On the contrary, more antagonistic behaviour is related to a less costly departure of the parent from their original preferences in order to spite the descendant.

## 4.3 Examples

In this section we present a few examples of altruistic preference functions. Clearly, conformism (recall Example 2) is altruistic, since it is more altruistic than its anchor $\mathcal{I}$. In contrast, contrarianism (recall Example 3) is antagonistic. Also, it follows directly from the definition that any paternalistic/egocentric preference function is both altruistic and antagonistic. In fact, this property characterises paternalism.

Below we address absolute altruism within Borda externality functions.

**Proposition 11.** *Any preference function $A : \mathscr{P} \to \mathscr{P}$ defined as in Examples 6, for some aggregator $f : X \times [0,1] \to \mathbb{R}$, is altruistic if it is anchored to some $\trianglerighteq \in \mathscr{P}$ and, for any $x \in X$, $m' \geq m$ implies $f(x, m') \geq f(x, m)$.*

*Proof.* Take any $\succeq \in \mathscr{P}$ and $x, x' \in X$ such that $x' \succeq x$ and $x' \trianglerighteq x$. Denote $m = \mu\big(L_{\succeq}(x)\big)$ and $m' = \mu\big(L_{\succeq}(x')\big)$, hence, we have $m' \geq m$. By assumption, it must be that $f(x, m') \geq f(x, m)$. Given that $A$ is anchored to $\trianglerighteq$, $x' \trianglerighteq x$ implies $f(x', m') \geq f(x, m')$. By combining these two inequalities with definitions of $m$ and $m'$, we obtain

$$f\Big(x', \mu\big(L_{\succeq}(x')\big)\Big) \;\geq\; f\Big(x, \mu\big(L_{\succeq}(x)\big)\Big).$$

Therefore, it must be that $x' A(\succeq) x$, which completes our proof. □

An immediate corollary is that the Borda rule is an altruistic function. This is because it is a special case of Borda externality, where $f(x, m) = \mu\big(L_{\trianglerighteq}(x)\big) + m$.

Monotonicity of the aggregator $f$ with respect to the second argument is not necessary for altruism. For example, suppose that $X = \{x, y\}$ and the function $f : X \times [0,1] \to \mathbb{R}$ satisfies $f(x, 1/2) > f(x, 1) > f(y, 1/2) > f(y, 1)$. Clearly, the function does not increase in the second argument. However, the preference function $A$, given by

$$z' A(\succeq)\, z \;\text{ if }\; f\Big(z', \mu\big(L_{\succeq}(z')\big)\Big) \;\geq\; f\Big(z, \mu\big(L_{\succeq}(z)\big)\Big),$$

for a uniform measure $\mu$, is anchored to $x \trianglerighteq y$ and is altruistic. Monotonicity is not necessary precisely because the measure $\mu\big(L_{\succeq}(z)\big)$ may never take different values for the same alternative $z$. As a result, when generating the ranking $A(\succeq)$, the violations of monotonicity $f(x, 1/2) > f(x, 1)$ and $f(y, 1/2) > f(y, 1)$ are never observed directly.

The above fact is no longer true if a preference function is defined over a richer space of alternatives $X$ and preferences $\mathscr{P}$. In Proposition A.6 of the Appendix, we show that in such a case, monotonicity of the function $f$ with respect to the second argument does become necessary for the corresponding preference function to be altruistic.

## 4.4 Absolute altruism and fixed points

We conclude Section 4 with a discussion on the existence and number of fixed points pertaining to altruistic and antagonistic preference functions.

**Corollary 2.** *Consider a preference function $A : \mathscr{P} \to \mathscr{P}$ that is anchored to some $\trianglerighteq \in \mathscr{P}$. The following statements are true.*

*(i) $A$ is altruistic only if $A(\trianglerighteq) = \trianglerighteq$.*

*(ii) $A$ is antagonistic and has a fixed point $\succeq$ only if $\succeq = \trianglerighteq$.*

*Proof.* Statement (i) follows immediately from Theorem 2(i), which requires that $x' \trianglerighteq (\triangleright)\, x$ implies $x' \trianglerighteq_A (\triangleright_A)\, x$, where $\trianglerighteq_A := A(\trianglerighteq)$. To prove statement (ii), suppose that $A$ is antagonistic and $A(\succeq) = \succeq$, for some $\succeq \in \mathscr{P}$. We claim that $\succeq = \trianglerighteq$. Towards contradiction, suppose that $x' \succeq x$ and $x \triangleright x'$, for some $x, x' \in X$. However, by statement (ii) in Theorem 2, $x' \succeq x$ and $x' A(\succeq) x$ implies $x' \trianglerighteq x$, yielding a contradiction. Similarly, it can never be that $x' \succ x$ and $x \trianglerighteq x'$. Therefore, we have $\succeq = \trianglerighteq$. □

For any altruistic function, the core preferences $\trianglerighteq$ are its fixed point. That is, whenever the parent interacts with a descendant who shares their core preferences, they follow them exactly, since there is no conflict of interests. Alternatively, an antagonistic preference function has at most one fixed point, which is always equal to $\trianglerighteq$.

Altruistic preferences exhibit a much stronger property of global convergence to a fixed point, as highlighted in the result below.

**Proposition 12.** *Suppose that $X$ is finite and the preference function $A : \mathscr{P} \to \mathscr{P}$ is anchored to some $\trianglerighteq \in \mathscr{P}$ and altruistic. For any $\succeq \in \mathscr{P}$, $A^n(\succeq)$ is a fixed point of $A$ for $n$ high enough, where we denote $A^n(\succeq) := \underbrace{A \circ A \circ \ldots \circ A}_{n \text{ times}}(\succeq)$.*

We postpone the proof until the Appendix. In general, altruistic preference functions may have multiple fixed points, even if they are anchored to a unique $\trianglerighteq$. For example, whenever $\mathcal{I} \in \mathscr{P}$, the conformistic function (recall Example 2) is altruistic and uniquely anchored to $\trianglerighteq = \mathcal{I}$, yet any $\succeq \in \mathscr{P}$ is its fixed point. Nevertheless, there is a wide class of functions that admit exactly one fixed point.

**Proposition 13.** *Suppose that $X = \mathbb{R}^\ell_+$ and $\mathscr{P}$ consists of continuous and (weakly) increasing preferences $\succeq$ over $X$.[12] Whenever $A : \mathscr{P} \to \mathscr{P}$ is altruistic and anchored to a strictly increasing preference relation $\trianglerighteq$, then $\trianglerighteq$ is the unique fixed point of $A$.*

*Proof.* Towards contradiction, suppose there is $\succeq \in \mathscr{P}$ such that $\succeq \neq \trianglerighteq$ and $A(\succeq) = \succeq$. First we show that there always exist some $x, x' \in X$ such that $x' \sim x$ and $x' \triangleright x$.

[12] We say that a binary relation $\succeq$ over $X = \mathbb{R}^\ell_+$ is (weakly) *increasing* if $x' \geq x$ implies $x' \succeq x$. The relation is *strictly increasing* if, in addition, $x' > x$ implies $x' \succ x$.

Since $\succeq \neq \trianglerighteq$, there must be some $y, y' \in X$ such that $y' \succeq y$ and $y \trianglerighteq y'$, where at least one of the comparisons is strict. Since both $\succeq$ and $\trianglerighteq$ are increasing, it must be that $y' \not\geq y$. Consider two cases. If (a) $y' \leq y$, then $y \sim y'$ and $y \triangleright y'$, thus, proving our claim for $x' = y$ and $x = y'$. Whenever (b) $y, y'$ are unordered, then $y' \succeq y \succeq y \wedge y'$.[13] Take any $\lambda \in [0,1]$ such that $y_\lambda := \left[\lambda y' + (1-\lambda) y \wedge y'\right] \sim y$. Since $y' > y_\lambda$ and $\trianglerighteq$ is strictly increasing, we have $y \trianglerighteq y' \triangleright y_\lambda$. Hence, our claim holds for $x = y_\lambda$ and $x' = y$.

To complete the proof, take any $x, x' \in X$ such that $x' \sim x$ and $x' \triangleright x$. Since $A$ is altruistic for $\trianglerighteq$, $x' \sim x$ and $x' \triangleright x$ implies $x' \succ_A x$, contradicting $A(\succeq) = \succeq$. □

Regarding antagonistic preferences, note that the contrarian function (recall Example 3) has a unique fixed point equal to $\mathcal{I}$. However, not every antagonistic function may admit a fixed point, as shown in the example below.

**Example 9.** Consider a weighted Borda rule function $A^\alpha$ introduced in Example 5', for some $\trianglerighteq \in \mathscr{P}$. Whenever $\alpha \leq 0$, the function $A^\alpha$ is antagonistic. In fact, as long as $\alpha > -1$, the unique fixed point of the function is $\trianglerighteq$. However, as the parent becomes more antagonistic and $\alpha \leq -1$, we have $x' A^\alpha(\trianglerighteq) x$ if, and only if, $x \trianglerighteq x'$.

We conclude this subsection by pointing out that a superposition of two altruistic functions, anchored to the same preference is also altruistic.

**Corollary 3.** *For any two altruistic functions $A, A' : \mathscr{P} \to \mathscr{P}$ anchored to the same preference relation $\trianglerighteq \in \mathscr{P}$, their superposition $A \circ A'$ is altruistic and anchored to $\trianglerighteq$.*

*Proof.* Take any $x, x' \in X$ and $\succeq \in \mathscr{P}$ such that $x' \succeq x$ and $x' \trianglerighteq x$. Since $A'$ is altruistic, it must be that $x' A'(\succeq) x$, by Theorem 2. Similarly, $x' A'(\succeq) x$ and $x' \trianglerighteq x$ implies that $x' A \circ A'(\succeq)\, x$, while $x' \triangleright x$ and $x' A'(\succeq) x$ implies $x' \succ_{A \circ A'} x$. □

The above observation is not true for altruistic functions that are anchored to different preferences $\trianglerighteq \neq \trianglerighteq'$, as shown in the example below.

**Example 10.** Let $X = \{x, y, z\}$ and $A, A'$ be two Borda rules defined as in Example 5 for the relations $x \triangleright y \triangleright z$ and $y \triangleright' z \triangleright' x$, respectively. As it was highlighted earlier, both functions are altruistic. We claim that $A \circ A'$ is *not* altruistic. First, note that $(A \circ A')(\mathcal{I}) = \succeq$, where $y \succ x \succ z$. Hence, we have $x \mathcal{I} y$ and $y \succ x$. However, we have

[13] By $y \wedge y'$, we denote the meet of $y$ and $y'$, where $(y \wedge y')_i = \min\{y_i, y'_i\}$, for all $i = 1, \ldots, \ell$.

$x \sim_{A \circ A'} y \succ_{A \circ A'} z$, where we denote $\succeq_{A \circ A'} := A \circ A'(\succeq)$. Therefore, we have $y \succ x$ and $x(A \circ A')(\succeq)y$. By Proposition 10(i), this contradicts that $A \circ A'$ is altruistic.

A composition of two antagonistic preference functions is, in general, *not* antagonistic, even if anchored to the same preference. For example, a superposition of a contrarian function with itself is equal to the conformistic function, which is altruistic.

# 5 Applications

In this section we employ our main results to specific economic applications. First, we revisit the problem of charitable giving. Then, we address the question of altruism and pocket money in multi-dimensional consumption spaces.

## 5.1 Charitable giving

Consider a dictator game as in Forsythe et al. (1994), Andreoni and Miller (2002), or Cox et al. (2008), among others. The parent divides a single good (e.g., money) between themselves and the descendant. Let $x_P \in X_P = \mathbb{R}_+$ denote the amount of the good kept by the parent, and $x_D \in X_D = \mathbb{R}_+$ be the amount given away to the descendant. We assume that $\mathscr{P}$ consists of upper semi-continuous preferences over $X = X_P \times X_D$.

To make our example consistent with the existing literature, we focus on choices from (possibly non-linear) budget sets. Formally, we say that $B \subseteq X$ is a *budget set*, if it is compact and satisfies $B = \{x \in X : g(x) \leq 0\}$, for some strictly increasing and continuous function $g : X \to \mathbb{R}$. For example, this holds for any linear budget set, with $g(x) = p \cdot x - m$, for prices $p \gg 0$ and income $m \geq 0$.

The main question we address in this section is: *Does more altruism lead to higher donations $x_D$?* By definition, choices of a more altruistic parent are always preferable to the descendant. Hence, whenever preferences of the descendant depend only on the received donation $x_D$ and are strictly increasing in it, the answer to our question is immediate. In this subsection we push this result further and argue that more altruism leads to higher donations, *regardless* of preferences of the descendant. In particular, this applies to descendants whose preferences depend non-trivially on payoffs $x_P$ of the parent, e.g., due to inequality aversion.

In order to make our analysis applicable to charitable giving, we focus on the class of preference functions that are anchored to the relation $\trianglerighteq$:

$$(x'_P, x'_D) \trianglerighteq (x_P, x_D) \quad \text{if, and only if,} \quad x'_P \geq x_P. \tag{1}$$

Thus, whenever it does not affect the welfare of the descendant, the parent would always prefer the alternative that yields them more money. Moreover, we shall restrict our attention to functions $A$ that take locally non-satiated values $A(\succeq)$, for all $\succeq \in \mathscr{P}$. We denote the space of all such preference functions by $\mathscr{A}_{\trianglerighteq}$.

**Proposition 14.** *Suppose that $A, A' \in \mathscr{A}_{\trianglerighteq}$. If $A$ and $A'$ are altruistic (antagonistic) and $A'$ is more altruistic (less altruistic) than $A$ then, for any $\succeq \in \mathscr{P}$, any budget set $B$, and any $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$ and $(x'_P, x'_D) \in \Phi_{A'(\succeq)}(B)$, there is $(y_P, y_D) \in \Phi_{A(\succeq)}(B)$ and $(y'_P, y'_D) \in \Phi_{A'(\succeq)}(B)$, such that $x_D \leq y'_D$ and $y_D \leq x'_D$.*

*Proof.* To prove the result outside of the parentheses, suppose that $A, A'$ are altruistic, and that $A'$ is more altruistic than $A$. Take any $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$. Towards contradiction, suppose that $y'_D < x_D$, for all $(y'_P, y'_D) \in \Phi_{A'(\succeq)}(B)$. Since $A(\succeq)$ and $A'(\succeq)$ are locally non-satiated and $B$ is a budget set, this implies that $y'_P > x_P$ and, thus, $(y'_P, y'_D) \triangleright (x_P, x_D)$. Moreover, by Theorem 1(v), we have $(y'_P, y'_D) \succeq (x_P, x_D)$. Otherwise, it would have to be that $(x_P, x_D) \in \Phi_{A'(\succeq)}(B)$, contradicting our initial assumption. However, since $A$ is altruistic, $(y'_P, y'_D) \succeq (x_P, x_D)$ and $(y'_P, y'_D) \triangleright (x_P, x_D)$ implies $(y'_P, y'_D) \succ_A (x_P, x_D)$, where $\succeq_A := A(\succeq)$ (by Theorem 2), which contradicts that $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$. We prove the second half of the result analogously.

Next, we prove the result within the parentheses. Suppose that $A, A'$ are antagonistic, and that $A$ is more altruistic than $A'$. Take any $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$. Towards contradiction, suppose that $y'_D < x_D$, for all $(y'_P, y'_D) \in \Phi_{A'(\succeq)}(B)$. As previously, this implies that $y'_P > x_P$ and, thus $(y'_P, y'_D) \triangleright (x_P, x_D)$. Moreover, as $A$ is more altruistic than $A'$, we have $(x_P, x_D) \succeq (y'_P, y'_D)$. Otherwise, by Theorem 1(v), it would have to be that $(x_P, x_D) \in \Phi_{A'(\succeq)}(B)$, yielding a contradiction. However, since $A$ is antagonistic, $(x_P, x_D) \succeq (y'_P, y'_D)$ and $(x_P, x_D) A(\succeq) (y'_P, y'_D)$ implies $(x_P, x_D) \trianglerighteq (y'_P, y'_D)$, which leads to a contradiction. We show the second part of the result analogously. □

This proposition requires some comment. When considering altruistic preference functions, the more altruistic parent always donates more to the descendant than the less altruistic one. However, for antagonistic functions the comparative statics are reversed—the less altruistic (more antagonistic) parent donates more. Although this result seems counterintuitive, the motive for which altruistic and antagonistic parents donate is different. For an altruistic parent, higher donations aim at improving the welfare of the descendant. For the antagonistic one, higher donations aim to spite the descendant. They donate more only when it makes the recipient worse off, e.g., due to guilt.

It is worth mentioning that our result specifies comparative statics, conditional on preferences of the descendant. That is, given $\succeq$, the donation of the more (less) altruistic parent is higher. However, depending on the preference itself, the donation may vary, as its size is conditional on the extent to which they affect welfare of the descendant.

The two observations above highlight that, without taking into account preferences of the descendant, defining more altruistic behaviour through higher donations could be misleading. This is precisely what is considered in Cox et al. (2008), where more altruistic behaviour is essentially identified through higher donation by the donor. Our result shows that, unless welfare of the recipient is taken into account, the size of donations may be a poor proxy for altruism and less altruistic parents may donate less.

Clearly, a more antagonistic parent donates more only when higher donation would make the descendant worse off. This could occur, e.g., when the descendant experiences an overwhelming guilt when the donation is too high. However, in many applications such preferences may not be considered relevant. In the following result we propose a class of preferences for which an antagonistic parent always donates nothing.

**Corollary 4.** *Let $A \in \mathscr{A}_{\trianglerighteq}$ be antagonistic. For any budget set $B$ and $\succeq\, \in \mathscr{P}$ such that $x_P > x'_P$ and $x'_D > 0$ implies $(x'_P, x'_D) \succeq (x_P, 0)$, if $(y_P, y_D) \in \Phi_{A(\succeq)}(B)$ then $y_D = 0$.*

*Proof.* Suppose that $y_D > 0$, for some $(y_P, y_D) \in \Phi_{A(\succeq)}(B)$. Since $A(\succeq)$ is locally non-satiated and $B$ is a budget set, there is some $(y'_P, 0) \in B$ such that $y'_P > y_P$. Therefore, by assumption, we have $(y_P, y_D) \succeq (y'_P, 0)$ and $(y_P, y_D) A(\succeq) (y'_P, 0)$, which implies that $(y_P, y_D) \trianglerighteq (y'_P, 0)$, since $A$ is antagonistic (recall Theorem 2). However, given that $y'_P > y_P$ implies $(y'_P, 0) \triangleright (y_P, y_D)$, this yields a contradiction. □

Whenever preferences of the descendant satisfy the condition stated in the corollary, an antagonistic parent would always donate 0, as this is the most severe way to hurt the descendant. The class of such preferences is quite broad. For example, the inequality aversion preferences as in Fehr and Schmidt (1999), induced by the utility function

$$u(x_P, x_D) \;=\; x_D - \alpha \max\big\{x_P - x_D, 0\big\} - \beta \max\big\{x_D - x_P, 0\big\},$$

satisfies the required condition for any $\alpha \geq 0$ and $\beta \in [0,1]$. However, the condition is violated whenever $\beta > 1$. In such a case, a spiteful parent may over-donate in order to hurt the descendant through guilt. However, following the empirical estimates in Bellemare et al. (2008), such cases are rather unlikely.

## 5.2 Pocket money problem

In our second application, we turn to donations in a multi-dimensional setting and discuss a problem of a parent who needs to decide how much pocket money to leave to the descendant. Let $x_P \in X_P = \mathbb{R}^\ell_+$ denote the consumption that is determined by the parent. It may include goods that are consumed privately, as well as public goods shared with the descendant. By $x_D \in X_D = \mathbb{R}^n_+$ we denote the private consumption of the descendant. Let $X = X_P \times X_D$ represent the consumption space. Throughout this section, we assume that $\mathscr{P}$ consists of upper semi-continuous preferences over $X$.

The parent is endowed with income $m$ that can be spent on public consumption $x_P$ at the prevailing prices $p_P \in \mathbb{R}^\ell_{++}$ and/or donated to the descendant as pocket money $q \in \mathbb{R}_+$, which the descendant will be able to spend on their own consumption $x_D$, given prices $p_D \in \mathbb{R}^n_{++}$. That is, given $q$ and $x_P$, the descendant maximises their preferences $\succeq$ over $C(x_P, q) := \big\{(x_P, x_D) : p_D \cdot x_D \leq q\big\}$. The objective of the parent is to maximise their own preferences $A(\succeq)$ with respect to $(x_P, x_D)$ and $q$ over

$$B := \Big\{(x_P, x_D) : p_P \cdot x_P + q \leq m \text{ and } (x_P, x_D) \in \Phi_{\succeq}\big(C(x_P, q)\big), \text{ for } q \geq 0\Big\}, \qquad (2)$$

where we identify the value of $q$ with the pocket money. In this subsection we are interested in the following question: *Under what conditions does the more altruistic parent give more pocket money $q$ to the descendant?*

Take any locally non-satiated preference relation $\trianglerighteq \in \mathscr{P}$ such that

$$(x_P, x'_D) \trianglerighteq (x_P, x_D) \quad \text{and} \quad (x_P, x_D) \trianglerighteq (x_P, x'_D),$$

for any $x_P \in X_P$ and $x_D, x'_D \in X_D$. That is, the ranking over $X_P$ is independent of the descendant's consumption. We restrict attention to preference functions $A$ that are anchored to $\trianglerighteq$. Moreover, we assume that values $A(\succeq)$ are locally non-satiated, over $X_P$, for all $\succeq \in \mathscr{P}$. That is, for any $(x_P, x_D)$, there is some arbitrarily close $(x'_P, x_D)$, such that $(x'_P, x_D) \succ (x_P, x_D)$. Denote the space of all such functions $A$ by $\mathscr{A}_{\trianglerighteq}$.

We say that preferences $\succeq \in \mathscr{P}$ are *consistent* with $\trianglerighteq$ over $X_P$, whenever $(x'_P, x_D) \trianglerighteq (x_P, x_D)$ implies $(x'_P, x_D) \succeq (x_P, x_D)$, for any $x_P, x'_P \in X_P$, $x_D \in X_D$. That is, conditional on $x_D$, both the parent and the descendant agree on the consumption of the public good $x_P$. In particular, this is satisfied for preferences $\succeq \in \mathscr{P}$ that are independent of $x_P$, i.e., satisfy $(x_P, x_D) \sim (x'_P, x_D)$, for all $x_P, x'_P \in X_P$ and $x_D \in X_D$.

**Proposition 15.** *Suppose that $A, A' \in \mathscr{A}_{\trianglerighteq}$ are altruistic. If $A'$ is more altruistic than $A$ then, for any $\succeq \in \mathscr{P}$ that is consistent with $\trianglerighteq$, and any $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$, $(x'_P, x'_D) \in \Phi_{A'(\succeq)}(B)$ with the corresponding pocket money $q, q'$, there is $(y_P, y_D) \in \Phi_{A(\succeq)}(B)$ $(y'_P, y'_D) \in \Phi_{A'(\succeq)}(B)$ with the corresponding pocket money $r, r'$, such that $q \leq r'$ and $r \leq q'$, where $B$ is given as in (2).*

*Proof.* First, we claim that, for any $\succeq \in \mathscr{P}$, the set $B$ is compact. Indeed, note that $B$ is the image of the correspondence $C$ over the set $D := \big\{(x_P, q) : p_P \cdot x_P + q \leq m\big\}$. Since $\succeq$ is upper semi-continuous and the set $\big\{(x_P, x_D) : p_D \cdot x_D \leq q\big\}$ is compact, the correspondence $(x_P, q) \to C(x_P, q)$ is compact-valued and upper hemi-continuous. By compactness of $D$, $B$ is compact (see Lemma 17.8 in Aliprantis and Border, 2006).

We proceed with proving the first part of the result. Take any $\succeq$ and $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$ with the corresponding pocket money $q$. Towards a contradiction, suppose that $q > r'$, for all $(y'_P, y'_D) \in \Phi_{A'(\succeq)}(B)$ and the corresponding pocket money $r'$. By local non-satiation, it must be that $q = p_D \cdot x_D$ and $r' = p_D \cdot y'_D$. Since $A'$ is more altruistic than $A$, we have $(y'_P, y'_D) \succ (x_P, x_D)$. Otherwise, by Proposition 2(v), it would have to be $(x_P, x_D) \in \Phi_{A'(\succeq)}(B)$, contradicting our initial assumption.

We claim that $(x_P, x_D) \trianglerighteq (y'_P, y'_D)$. Indeed, since $(x_P, x_D) \notin \Phi_{A'(\succeq)}(B)$, we have $(y'_P, y'_D) \succ_{A'} (x_P, x_D)$. Given that $A, A'$ are altruistic, Theorem 2 would require that $(y'_P, y'_D) \triangleright (x_P, x_D)$ implies $(y'_P, y'_D) \succ_A (x_P, x_D)$ and, thus, contradicting that $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$. As $\trianglerighteq$ is independent of the consumption of the descendant, it follows that $(x_P, y'_D) \trianglerighteq (y'_P, y'_D)$. By consistency of $\succeq$ with $\trianglerighteq$, we have $(x_P, y'_D) \succeq (y'_P, y'_D)$.

To conclude the first part of the proof, note that $q = p_D \cdot x_D > p_D \cdot y'_D = r'$ implies $(y'_P, y'_D) \succ (x_P, x_D) \succeq (x_P, y'_D)$. However, this contradicts that $(x_P, y'_D) \succeq (y'_P, y'_D)$. The second part of the result can be shown analogously. □

Whenever preferences of the parent and the descendant are consistent over $X_P$, the more altruistic parent always devotes more of the budget to the consumption of the descendant. Therefore, more altruistic parents give more pocket money. The consistency between the two preferences is critical. Otherwise, it would be possible for the more altruistic parent to give less pocket money, but compensate the loss in welfare of the descendant with a choice of a public good that would be more favourable to the latter. The consistency condition excludes this possibility.

Unlike in charitable giving, giving no pocket money is always optimal for an antagonistic parent. By the nature of the problem, the descendant would always (weakly) prefer more pocket money to less, regardless of their preferences. Hence, donating zero to the descendant is the most spiteful behaviour available in this setting. As a result, the behaviour of one antagonistic parent is indistinguishable from another.

**Proposition 16.** *Take any $A \in \mathscr{A}_{\trianglerighteq}$. If $A$ is antagonistic then, for any $\succeq \in \mathscr{P}$, $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$ implies $p_D \cdot x_D = q = 0$.*

*Proof.* Towards a contradiction, suppose there is some $(x_P, x_D) \in \Phi_{A(\succeq)}(B)$ such that $p_D \cdot x_D > 0$. Therefore, it must be that $(x_P, x_D) \succeq (x_P, 0)$ and $(x_P, x_D) \succ_A (x_P, 0)$, by local non-satiation of $A(\succeq)$. Since $A$ is antagonistic, this implies $(x_P, x_D) \triangleright (x_P, 0)$. However, by assumption, we have $(x_P, 0) \trianglerighteq (x_P, x_D)$, yielding a contradiction. $\square$

# A Appendix

Here we present the proofs omitted from the main body of the paper along with some auxiliary results. First, we discuss the extension to general spaces of the class of Borda externality preference functions introduced in Example 6.

## A.1 Borda externality in general spaces

Before presenting the generalised version of Borda externality, we need to introduce some initial results regarding the representation of preferences using measures.

**Proposition A.1.** *Let $X$ be a topological space and $\succeq$ be an upper semi-continuous preference relation.*[14] *For any finite measure $\mu$ over the Borel sigma-algebra of $X$, the*

[14] A preference relation $\succeq$ over a topological space $X$ is *upper semi-continuous*, if the strict lower contour set $L^\circ_\succeq(x)$ is open, for all $x \in X$.

*function* $\bar{\mu}_{\succeq} : X \to \mathbb{R}$*, given by* $\bar{\mu}_{\succeq}(x) := \inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big)$*, is well-defined, upper semi-continuous, and* $x' \succeq x$ *implies* $\bar{\mu}_{\succeq}(x') \geq \bar{\mu}_{\succeq}(x)$*, i.e., it is a pseudo-utility.*

*Proof.* Since $\succeq$ is upper semi-continuous, the set $L^{\circ}_{\succeq}(y)$ is open, for all $x \in X$. Therefore, the function is well-defined. To show that it is upper semi-continuous, we claim that

$$\Big\{x \in X : \inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big) < \alpha\Big\}$$

is open, for any number $\alpha$. Whenever $\inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big) < \alpha$, there must be some $y \succ x$ such that $\mu\big(L^{\circ}_{\succeq}(y)\big) < \alpha$. In particular, $L^{\circ}_{\succeq}(y)$ is open and contains $y$. Moreover, for all $z \in L^{\circ}_{\succeq}(y)$, we have $\inf_{y \succ z} \mu\big(L^{\circ}_{\succeq}(y)\big) \leq \mu\big(L^{\circ}_{\succeq}(y)\big) < \alpha$, which completes the proof.

To show that $\bar{\mu}_{\succeq}$ is a pseudo-utility, take any $x, x' \in X$. Whenever $x \sim x'$, we have $\bar{\mu}_{\succeq}(x) = \bar{\mu}_{\succeq}(x')$, since $y \succ x$ is equivalent to $y \succ x'$, for all $y \in X$. Alternatively, suppose that $x' \succ x$. Since $L^{\circ}_{\succeq}(x') \subseteq \bigcap_{y \succ x'} L^{\circ}_{\succeq}(y)$, we obtain

$$\begin{aligned} \bar{\mu}_{\succeq}(x) \;&=\; \inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big) \;\leq\; \mu\big(L^{\circ}_{\succeq}(x')\big) \\ &\leq\; \mu\Big(\bigcap\nolimits_{y \succ x'} L^{\circ}_{\succeq}(y)\Big) \;\leq\; \inf_{y \succ x'} \mu\big(L^{\circ}_{\succeq}(y)\big) \;=\; \bar{\mu}_{\succeq}(x'). \end{aligned}$$

This completes our proof. □

Whenever $X$ is finite and $\mu$ has the full support, then the function $\bar{\mu}_{\succeq}$ represents any preference relation $\succeq$ over $X$ and $\bar{\mu}_{\succeq}(x) = \mu\big(\{y \in X : y \preceq x\}\big)$. Below, we show that the same property holds for any continuous preferences defined over a connected space.

**Proposition A.2.** *Suppose that* $X$ *is a connected topological space and* $\succeq$ *is a continuous preference relation. For any finite measure* $\mu$ *over the Borel sigma-algebra of* $X$*, we have* $\bar{\mu}_{\succeq}(x) = \mu\big(\{y \in X : y \preceq x\}\big)$*, for all* $x \in X$*, where* $\bar{\mu}_{\succeq}$ *is defined as in Proposition A.1. Moreover, if* $\mu$ *has full support, then* $\bar{\mu}_{\succeq}$ *is a utility representation of* $\succeq$*.*

*Proof.* We prove the first part of the result. Since $\{y \in X : y \preceq x\}$ is closed and $\mu$ is defined over the Borel sigma-algebra, the value $\mu\big(\{y \in X : y \preceq x\}\big)$ is well-defined. Moreover, we have $\{y \in X : y \preceq x\} = \bigcap_{z \succ x} L^{\circ}_{\succeq}(z) = \inf_{z \succ x} L^{\circ}_{\succeq}(z)$. Thus, we have

$$\bar{\mu}_{\succeq}(x) \;=\; \inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big) \;=\; \mu\big(\{y \in X : y \preceq x\}\big).$$

We prove the second part of the result. Given that $\bar{\mu}_{\succeq}$ is a pseudo-utility, it suffices to show that $x' \succ x$ implies $\bar{\mu}_{\succeq}(x') > \bar{\mu}_{\succeq}(x)$. Since $\succeq$ is continuous and $X$ is connected,

$$S \;=\; \big\{y \in X : y \succ x\big\} \cap \big\{y \in X : y \prec x'\big\}$$

is open and non-empty. Since $\mu$ has full support, this implies that

$$\bar{\mu}_{\succeq}(x) \;=\; \mu\big(\{y \in X : y \preceq x\}\big) \;<\; \mu\big(\{y \in X : y \preceq x\}\big) + \mu(S) \\ \leq\; \mu\big(\{y \in X : y \preceq x'\}\big) \;=\; \bar{\mu}_{\succeq}(x'),$$

which concludes our proof. □

Although Proposition A.2 provides conditions under which $\bar{\mu}_{\succeq}$ is a utility representation of a continuous preference relation, it does not guarantee that the function is continuous—only upper semi-continuous (see Proposition A.1). In fact, in general, the function $\bar{\mu}_{\succeq}$ is *not* continuous unless $\succeq$ has "thin" indifference curves. A version of the following result is stated in Neuefeind (1972). We state it for completeness.

**Proposition A.3.** *Let $X$ be a topological space and $\mu$ be a finite measure over the Borel sigma-algebra of $X$. If $\succeq$ is a continuous preference relation and $\mu\big(L_{\succeq}(x) \setminus L^{\circ}_{\succeq}(x)\big) = 0$, for all $x \in X$, the function $\bar{\mu}_{\succeq}$, defined in Proposition A.1, is continuous.*

*Proof.* It suffices to show that $\bar{\mu}_{\succeq}$ is lower semi-continuous. We claim that

$$S := \Big\{x \in X : \inf_{y \succ x} \mu\big(L^{\circ}_{\succeq}(y)\big) > \alpha\Big\}$$

is open, for any number $\alpha$. Take any $x \in S$. If there is some $x' \in S$ such that $x \succ x'$, then $x \in U^{\circ}_{\succeq}(x') \subseteq S$. Since $\succeq$ is continuous, $U^{\circ}_{\succeq}(x')$ is open. Conversely, suppose that $x \succeq x'$ implies $\bar{\mu}_{\succeq}(x') \leq \alpha$. Then it must be that $\alpha \geq \sup_{x' \succ x} \bar{\mu}_{\succeq}(x') = \mu\big(L^{\circ}_{\succeq}(x)\big) = \mu\big(L^{\circ}_{\succeq}(x)\big) + \mu\big(L_{\succeq}(x) \setminus L^{\circ}_{\succeq}(x)\big) = \mu\big(L_{\succeq}(x)\big) = \bar{\mu}_{\succeq}(x) > \alpha$, which yields a contradiction. □

We are ready to extend Example 6 to general spaces. Let $X$ be a topological space and $\mathscr{P}$ consist of upper semi-continuous preferences over $X$. For any probability measure $\mu$ over the Borel sigma-algebra of $X$, and any $\succeq \in \mathscr{P}$, define the function $\bar{\mu}_{\succeq} : X \to \mathbb{R}$ as in Proposition A.1. For any upper semi-continuous function $f : X \times [0,1] \to \mathbb{R}$ that is increasing in the second argument, the function $A : \mathscr{P} \to \mathscr{P}$, given by

$$x' A(\succeq)\, x \quad \text{if} \quad f\big(x', \bar{\mu}_{\succeq}(x')\big) \geq f\big(x, \bar{\mu}_{\succeq}(x)\big),$$

is well-defined. Indeed, given Proposition A.1, the function $x \to f\big(x, \bar{\mu}_{\succeq}(x)\big)$ is upper semi-continuous, for any $\succeq \in \mathscr{P}$. This is sufficient for the relation $A(\succeq)$ to be upper semi-continuous. Given that the function $f(x,m) := \bar{\mu}_{\succeq}(x) + m$ is upper semi-continuous and increasing in the second argument, the Borda rule (see Example 5) can be extended to more abstract spaces in the analogous way.

Next, we prove some basic properties of Borda externalities. First, we claim that, under certain conditions, any such function $A$ is anchored only if $f(x',m) \geq f(x,m)$ implies $f(x',m') \geq f(x,m')$, for any $x, x' \in X$ and $m, m' \in [0,1]$.

**Proposition A.4.** *Suppose that $X \subseteq \mathbb{R}^\ell$ is Euclidean, $\mu$ is a probability measure absolutely continuous with respect to the Lebesgue measure, and $\mathscr{P}$ contains all continuous preferences $\succeq$ such that $\mu\big(L_\succeq(x) \setminus L^\circ_\succeq(s)\big) = 0$, for all $x \in X$. Then, a Borda externality function $A : \mathscr{P} \to \mathscr{P}$ defined above is anchored only if $f(x',m) \geq f(x,m)$ implies $f(x',m') \geq f(x,m')$, for any $x, x' \in X$ and $m, m' \in [0,1]$.*

Before we prove the proposition, we need to introduce three auxiliary results.

**Lemma A.1.** *Let $X \subseteq \mathbb{R}^\ell$ be Euclidean and $\mu$ be a finite, non-atomic measure over the Borel sigma-algebra of $X$, that is absolutely continuous with respect to the Lebesgue measure. For any $x \in X$, the function $f : \mathbb{R}_+ \to [0,1]$, given by $f(r) := \mu\big(B(x,r)\big)$, is continuous, where $B(x,r)$ denotes the closed ball centred at $x$ with the radius $r$.*

*Proof.* Take any sequence $r^n$ that converges to some $r$ from above. Then $\lim_n f(r^n) = \lim_n \mu\big(B(x,r^n)\big) = \mu\big(\bigcap_n B(x,r^n)\big) = \mu\big(B(x,r)\big) = f(r)$. Whenever $r^n$ converges to $r$ from below, we have $\lim_n f(r^n) = \lim_n \mu\big(B(x,r^n)\big) = \mu\big(\bigcup_n B(x,r^n)\big) = \mu\big(B^\circ(x,r)\big) = \mu\big(B(x,r)\big) = f(r)$, where the equality $\mu\big(B^\circ(x,r)\big) = \mu\big(B(x,r)\big)$ follows from $\mu$ being absolutely continuous with respect to the Lebesgue measure. □

**Lemma A.2.** *Let $X \subseteq \mathbb{R}^\ell$ be Euclidean and $\mu$ be a finite, non-atomic measure over the Borel sigma-algebra of $X$, that is absolutely continuous with respect to the Lebesgue measure. For any $x \in X$ and $m \in [0,1)$, there is some $r$ such that $\mu\big(B(x,r)\big) = m$.*

*Proof.* Take any $x \in X$ and $m \in [0,1)$. By Lemma A.1, the function $f(r) := \mu\big(B(x,r)\big)$ is continuous, $f(0) = 0$, and $\lim_{r\to\infty} f(r) = 1$. By the intermediate value theorem, for any $m \in [0,1)$, there is some $r$ such that $f(r) = m$. □

The next result posits that the space of continuous preferences is sufficiently rich. That is, for any measure $\mu$ and alternatives $x, x' \in X$, one can always find a continuous preference relation with "thin" indifference curves for which the measure of the lower contour sets corresponding to $x$ and $x'$ takes an arbitrary value.

**Lemma A.3.** *Let $X \subseteq \mathbb{R}^\ell$ and $\mu$ be a finite, non-atomic measure over the Borel sigma-algebra of $X$, that is absolutely continuous with respect to the Lebesgue measure. For any $x, x' \in X$ and $m, m' \in [0,1]$, there is a continuous preference relation $\succeq$ over $X$ such that $\mu\big(L_\succeq(x)\big) = m$ and $\mu\big(L_\succeq(x')\big) = m'$. Moreover, $\succeq$ has "thin" indifference curves, i.e., we have $\mu\big(L_\succeq(x) \setminus L^\circ_\succeq(x)\big) = 0$, for all $x \in X$.*

*Proof.* Take any $x, x' \in X$ and $m, m' \in [0,1]$. Without loss of generality, suppose that $m' \geq m$. First, we show that there exist closed sets $S \subseteq S'$ such that $x \in S$, $x' \in S'$, and $\mu(S) = m$, $\mu(S') = m'$. Moreover, if $m' > m$, then $x' \notin S$.

Consider the following four cases. If (i) $m = m' = 1$, set $S' = X$. If (ii) $m = 0, m' = 1$, set $S' = X$ and $S = \{x\}$. If (iii) $1 = m' > m > 0$, let $S' = X$. Denote $\epsilon = m' - m < 1$. By Lemma A.2, there is some $r$ such that $\mu\big(B(x', r)\big) = \epsilon = \mu\big(B^\circ(x', r)\big)$, since $\mu$ is absolutely continuous with respect to the Lebesgue measure. Let $S := \big(X \setminus B^\circ(x', r)\big) \cup \{x\}$, which is closed, contains $x$, and satisfies $\mu(S) = 1 - \epsilon = m$.

Finally, suppose that (iv) $1 > m' \geq m > 0$. By Lemma A.2, there is some $r'$ such that $\mu\big(B(x', r')\big) = m'$. Let $S' = B(x', r') \cup \{x\}$. To construct the set $S$, denote $\epsilon = m' - m$. If $\epsilon = 0$, let $S = S'$. Otherwise, take any $r$ such that $\mu\big(B(x', r)\big) = \epsilon$. Since $\mu$ is absolutely continuous with respect to the Lebesgue measure, $\mu\big(B^\circ(x', r)\big) = \epsilon$. Define $S := \big(S' \setminus B^\circ(x', r)\big) \cup \{x\}$, which satisfies the required conditions.

We complete the proof by, essentially, applying the strong Urysohn's lemma. Define function $g : S \to [0,1]$ as follows. If the interior of $S$ is empty, set the $g(y) := 0$, for all $y \in S$. Otherwise, take any $z \in S^\circ$ that is different from $x$. Define

$$g(y) \;:=\; \frac{d(y,z)}{d(y,z) + d\big(y, \partial S \cup \{x\}\big)},$$

where $d$ denotes the distance function. In particular, the function is continuous with "thin" level curves, while $g(y) = 1$ is equivalent to $y \in \partial S \cup \{x\}$. Define the function $h : S' \to [0,2]$ as follows. If $y \in S$, let $h(y) := g(y)$; otherwise

$$h(y) \;:=\; 1 + \frac{d(y,S)}{d(y,S) + d\big(x, \partial S' \cup \{x'\}\big)}.$$

Note that the function is continuous with "thin" indifference curves, while $h(y) = 2$ if, and only if $y \in S' \cup \{x'\}$. Finally, we define the function $f : X \to [0,3]$. If $S' = X$, set $f(y) = h(y)$, for all $y \in X$. Otherwise, take any $z \in X \setminus S'$. If $y \in S'$, let $f(y) := h(y)$; otherwise, set $f(y) := 2 + d(y, S')/\big[d(y,S') + d(y,z)\big]$. It is easy to show that the function $f$ is continuous and has "thin" level curves.

To complete the proof, let $\succeq$ denote the preference relation induced by the function $f$. Clearly, it is continuous, has "thin" indifference curves, while $\mu\big(L_{\succeq}(x)\big) = \mu\big(\{y : f(y) \leq 1\}\big) = \mu(S) = m$ and $\mu\big(L_{\succeq}(x')\big) = \mu\big(\{y : f(y) \leq 2\}\big) = \mu(S') = m'$. $\square$

The proof of Proposition A.4 is immediate. Towards contradiction, suppose that $f(x', m) \geq f(x, m)$ and $f(x', m') < f(x, m')$, for some $x, x' \in X$ and $m, m' \in [0, 1]$. By Lemma A.3, there are preferences $\succeq, \succeq' \in \mathscr{P}$ such that $\mu\big(L_{\succeq}(x)\big) = \mu\big(L_{\succeq}(x')\big) = m$ and $\mu\big(L_{\succeq'}(x)\big) = \mu\big(L_{\succeq'}(x')\big) = m'$, and so $x \sim x'$ and $x \sim' x'$. Since $f\big(x', \mu\big(L_{\succeq}(x')\big)\big) \geq f\big(x, \mu\big(L_{\succeq}(x)\big)\big)$ implies $x' A(\succeq) x$, and $f\big(x', \mu\big(L_{\succeq'}(x')\big)\big) < f\big(x, \mu\big(L_{\succeq'}(x)\big)\big)$ implies *not* $x' A(\succeq') x$, this contradicts that $A$ is anchored.

Below, we claim that Borda externality preference function are ordered in the "more altruistic" sense only if they satisfy the single-crossing condition in Proposition 9.

**Proposition A.5.** *Let $X \subseteq \mathbb{R}^\ell$ be Euclidean, $\mu$ be a probability measure that is absolutely continuous with respect to the Lebesgue measure, and $\mathscr{P}$ contain all continuous preferences $\succeq$ such that $\mu\big(L_{\succeq}(x) \backslash L^{\circ}_{\succeq}(x)\big) = 0$, for all $x \in X$. For any two Borda externality preference functions $A, A' : \mathscr{P} \to \mathscr{P}$, defined for some aggregators $f, f' : X \times [0, 1] \to \mathbb{R}$, respectively, the function $A'$ is more altruistic than $A$ only if the condition in Proposition 9 holds.*

*Proof.* Towards contradiction, suppose that $A'$ is more altruistic than $A$, but there is some $x, x' \in X$ and $m, m' \in [0, 1]$ such that $f(x', m') \geq f(x, m)$ and $f'(x', m') < f'(x, m)$. By Lemma A.3, there is a preference relation $\succeq \in \mathscr{P}$ such that $\mu\big(L_{\succeq}(x')\big) = m' > m = \mu\big(L_{\succeq}(x')\big)$. In particular, it must be that $x' \succ x$. However, since $f\big(x', \mu\big(L_{\succeq}(x')\big) \geq f\big(x, \mu\big(L_{\succeq}(x)\big)\big)$ implies $x' A(\succeq) x$, and $f'\big(x', \mu\big(L_{\succeq}(x')\big)\big) < f'\big(x, \mu\big(L_{\succeq}(x)\big)\big)$ implies *not* $x' A'(\succeq) x$, this contradicts that $A'$ is more altruistic than $A$. Analogously, one can show that $f(x', m') > f(x, m)$ and $f'(x', m') \leq f'(x, m)$ leads to a contradiction as well. $\square$

We conclude this section by referring to Proposition 11 and showing that monotonicity of the aggregator $f$ with respect to the second variable is necessary for the corresponding preference function to be altruistic. Before stating the result, we say that the function $f$ is locally weakly non-satiated on $X$ if, for any $x \in X$, $m \in [0, 1]$, and an open set $U \ni x$, there is some $x' \in U$ such that $f(x', m) \geq f(x, m)$.

**Proposition A.6.** *Let $X \subseteq \mathbb{R}^\ell$ be Euclidean and $\mu$ be a finite, non-atomic measure over the Borel sigma-algebra of $X$, that is absolutely continuous with respect to the Lebesgue* *measure. Consider a function* $A : \mathscr{P} \to \mathscr{P}$ *given by*

$$x' A(\succeq) x \quad \text{if} \quad f\Big(x', \mu\big(L_{\succeq}(x')\big)\Big) \geq f\Big(x, \mu\big(L_{\succeq}(x)\big)\Big),$$

*for some function* $f : X \times [0,1] \to \mathbb{R}$ *that is continous and locally weakly non-satiated on* $X$*. The function* $A$ *is altruistic only if* $f(x, m') \geq f(x, m)$*, for all* $x \in X$ *and* $m' \geq m$*.*

*Proof.* Clearly, the condition must hold for $m = m'$. Towards contradiction, suppose that $f(x, m') < f(x, m)$, for some $x \in X$ and $m' > m$. Take any sequence of points $x^n$ converging to $x$ such that $f(x^n, m') \geq f(x, m')$, for all $n$. Since $f$ is locally weakly non-satiated on $X$, such a sequence exists. Moreover, by Proposition A.4, it must be that $x^n \trianglerighteq x$, for all $n$, where $\trianglerighteq$ denotes the relation to which $A$ is anchored.

Construct a sequence of preferences $\succeq^n \in \mathscr{P}$ such that $\mu\big(L_{\succeq^n}(x^n)\big) = m' > m = \mu\big(L_{\succeq^n}(x)\big)$, for all $n$. By Lemma A.3, such preferences exist. Moreover, it must be that $x^n \succeq^n x$, for all $n$. Whenever $A$ is altruistic, we have $x^n \trianglerighteq x$ and $x^n \succeq^n x$ only if $f(x^n, m') \geq f(x, m)$. However, since $f$ is continuous on $X$, as $x^n$ converges to $x$, it must be that $f(x, m') \geq f(x, m)$, yielding a contradiction. □

## A.2 Proof of Proposition 12

In this subsection we prove Proposition 12. First, however, we need to introduce some additional terminology and one auxiliary result. For any preference relations $\trianglerighteq$ and $\succeq, \succeq'$ in $\mathscr{P}$, we say that $\succeq'$ is *more consistent* with $\trianglerighteq$ than $\succeq$ if, for any $x, x' \in X$, $x' \trianglerighteq x$ and $x' \succeq x$ implies $x' \succeq' x$, and $x' \triangleright x$ and $x' \succ x$ implies $x' \succ' x$.

**Proposition A.7.** *The "more consistent" relation is a partial order.*

*Proof.* We need to show that the relation is transitive and antisymmetric. Take any relations $\succeq, \succeq', \succeq'' \in \mathscr{P}$ such that $\succeq''$ is more consistent with $\trianglerighteq$ than $\succeq'$, and $\succeq'$ is more consistent than $\succeq$. In particular, for any $x, x' \in X$, if $x' \trianglerighteq x$ and $x' \succeq x$, then $x' \succeq' x$. Moreover, $x' \trianglerighteq x$ and $x' \succeq' x$ implies $x' \succeq'' x$. Similarly, whenever $x' \triangleright x$ and $x' \succ x$, then $x' \succ' x$, while $x' \triangleright x$ and $x' \succ' x$ implies $x' \succ'' x$.

Next, we show that it is antisymmetric. Suppose that $\succeq$ is more consistent with $\trianglerighteq$ than $\succeq'$, and vice-versa. Towards contradiction, suppose that $\succeq \neq \succeq'$. In particular, there is some $x, x' \in X$ such that $x \succeq x'$ and $x' \succ' x$. Suppose that $x \trianglerighteq x'$. Then, $x \succeq x'$ implies $x \succeq' x'$, which yileds a contradiction. Conversely, assume that $x' \triangleright x$. However, in this case, $x' \succ' x$ implies $x' \succ x$. Hence, it must be that $\succeq = \succeq'$. □

We proceed with the proof of Proposition 12.

*Proof of Proposition 12.* First, we show that, for any $\succeq \in \mathscr{P}$, the relation $A(\succeq)$ is more consistent with $\trianglerighteq$ than $\succeq$. Indeed, by altruism, if $x' \succeq x$ and $x' \trianglerighteq x$ then $x' A(\succeq) x$, where the latter is strict whenever $x' \triangleright x$. This implies that the sequence $A^n(\succeq)$ is increasing in the "more consistent with" sense. Since $X$ is finite, so is $\mathscr{P}$. Therefore, the sequence $A^n(\succeq)$ reaches its maximal element, which is its fixed point. □

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