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Note

A note on Markov perfect equilibria in a class of non-stationary stochastic bequest games

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ABSTRACT

In this note, we prove the existence of a Markov perfect equilibrium in a non-stationary version of a paternalistic bequest game. The method we advocate is general and allows to study models with unbounded state space and unbounded utility functions. We cover both, the stochastic and deterministic cases. We provide a characterization of the set of all Markov perfect equilibria by means of a set-valued recursive equation involving the best response operator. In the stationary case, we show that there exists a set of strategies that is invariant under the best response mapping.

1. Introduction

Since the seminal papers of Phelps and Pollak [16], Kohlberg [11], Bernheim and Ray [7], or Leininger [12] researchers study a class of paternalistic bequest economies. Specifically, the economy they consider consists of a sequence of generations, each living one period, and deriving utility from its own consumption, as well as that of a successor generation. At the beginning of each period t, generation receives an endowment of a single homogeneous good, which for $t \geq 2$ is the output from a bequest left by the previous generation. This endowment is divided between consumption and investment.

From a game-theoretic point of view this economy is represented by an infinite horizon "bequest game" on an uncountable, possibly unbounded state space with countably many short-lived players (generations). A natural solution concept for this class of games is the subgame (or Markov) perfect equilibrium. First results on the existence of Markov perfect equilibria in bequest games with deterministic transitions were

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proved (by different methods) in [7] and [12]. Although the model of the bequest game looks simple, the proofs given in the aforementioned papers are quite involved and based on some technical tricks. A simpler and more transparent proof is given in [4].

Extensions of the simple bequest game model involving more than one descendant for each generation and stochastic transitions were considered by many authors. A survey of the existing literature (both theoretical results and applications in economics) can be found in [10] and [6]. In the stochastic framework, the most general results (from the point of view of assumptions on the transition structure) are given in [2,3].

It should be noted that almost all of the papers in the literature were devoted to the stationary models where utility functions and transitions are independent of time. The authors have focused on existence and computation of (Markov) stationary Nash equilibria due to their simplicity and computational tractability. General techniques and results concerning *non-stationary* equilibria or *non-stationary* games are not very well developed. That is, while the existence of a stationary (Markov) perfect equilibrium in a stationary intergenerational game is a fixed point problem of a best response mapping in an appropriately defined function space, characterizations of the sets of non-stationary Markov perfect equilibria in bequest games are almost not known in the existing literature. Assuming that the utility functions and transitions are independent of generation number is obviously a restriction. It is natural to expect that tastes and production technologies change in time.

Existence of Markov perfect equilibria in a non-stationary deterministic game with bounded state space was established by Bernheim and Ray [7]. However, it is not clear how to extend the results given in [7] to an unbounded state space case. A natural idea to construct sets of Markov perfect equilibria in non-stationary bequest games is to use a generalization of strategic dynamic programming arguments leading to some "set-valued recursive equation." Similar methods were used by Harris [9] for perfect equilibria in games of perfect information and Mertens and Parthasarathy [13] in their study of subgame perfect equilibria in discounted stochastic games as well as by Abreu et al. [1] in repeated games. Specifically, instead of analyzing fixed points of best response maps defined on particular classes of strategies, they construct a family of descending self-generating subsets of the strategies or the value sets, and show that these sets have a non-empty intersection. Then, they select a sequence of strategies (or values) from this intersection (the limiting set) and obtain the desirable equilibrium solution. For a short introduction and sketch of the main arguments used in studying equilibria in standard stochastic games the reader is also referred to pages 397–398 in [14]. A suitable modification of the Mertens and Parthasarathy method [13] suggests a direct way to construct the equilibrium sets in some classes of dynamic games with quasi-hyperbolic discounting as in [5].

In this paper, we apply the Mertens and Parthasarathy type method [13] to characterize the sets of non-stationary Markov perfect equilibria in a large class of infinite horizon bequest games with both stochastic and deterministic transitions. As corollaries we prove the existence of non-stationary Markov perfect equilibrium in two classes of games with different utility functions. Our approach is general and allows to analyze unbounded state spaces, unbounded utility functions as well as both, the stochastic and deterministic cases. To the best of our knowledge, existence of an MPE in non-stationary stochastic bequest games has not been established in the literature so far.

Additional and more detailed comments on our results and the literature are given in the last section of this note.

2. Non-stationary stochastic bequest games with additive utilities

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all positive integers. Let $S := [0, +\infty)$, $S_+ := (0, +\infty)$ and $A(s) := [0, s]$ for $s \in S$. Consider an infinite sequence of generations labeled by $t \in \mathbb{N}$. There is one commodity, which may be consumed or invested. Every generation lives one period and derives utility from its own consumption and consumption of its immediate descendant. Generation $t \in \mathbb{N}$ receives the endowment $s_t \in S$ and chooses a consumption level $a_t \in A(s_t)$. The investment of $y_t := s_t - a_t$ determines the endowment of its successor according to some transition probability q_t from S to S, which depends on $y_t \in A(s_t)$. If $s_t = s$, then we shall often write s' for s_{t+1}.

Let Φ be the set of Borel measurable functions $\phi : S \to S$ such that $\phi(s) \in A(s)$ for each $s \in S$. A *strategy* for generation $t \in \mathbb{N}$ is a function $c_t \in \Phi$.

The transition probability induced by q_t and $c_t \in \Phi$ is $q_t(\cdot|i_t(s))$, where $i_t(s) := s - c_t(s)$ is the *investment* or *saving* in state $s \in S$. Assume that generation $t \in \mathbb{N}$ consumes $a \in A(s_t)$ in state $s_t = s$ and the following generation is going to use a strategy $c_{t+1} \in \Phi$. Then, the *expected utility* of generation t is defined as follows

$$P_t(a, c_{t+1})(s) := u_t(a) + \int_S v_t(c_{t+1}(s'))q_t(ds'|s - a), \tag{1}$$

where $u_t : S \to S$ whereas $v_t : S \to S$ is bounded and Borel measurable. In the sequel, we impose additional assumptions on functions u_t and v_t.

The solution concept given below dates back to Phelps and Pollak [16].

Definition 1. A *Markov Perfect Equilibrium* (*MPE*) in the model with the expected utility defined in (1) is a sequence of strategies $(c_t^*)_{t\in\mathbb{N}}$ such that $c_t^* \in \Phi$ for each $t \in \mathbb{N}$ and

$$\sup_{a\in A(s)} P_t(a, c_{t+1}^*)(s) = P_t(c_t^*(s), c_{t+1}^*)(s)$$

for every $s \in S$ and $t \in \mathbb{N}$. An *MPE* $(c_t^*)_{t\in\mathbb{N}}$ is *stationary*, if $c_t^* = c_1^*$ for each $t \in \mathbb{N}$.

Hence, an *MPE* is a sequence (c_t^*) of measurable functions mapping current states to consumption choices such that, if generation $t+1$ is going to use c_{t+1}^*, then the best response of generation t is to use c_t^*. Observe that (c_t^*) is a Nash equilibrium in the game played by countably many players (generations) using Markov strategies.

Let $C(S)$ be the set of all bounded real-valued continuous functions on S. By $P(S)$ we denote the set of all probability measures on the Borel subsets of S. We endow $P(S)$ with the topology of weak convergence. Recall that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on S *converges weakly* to some $\mu \in P(S)$, if for every $g \in C(S)$, it holds that

$$\lim_{n\to\infty} \int_S g(s)\mu_n(ds) = \int_S g(s)\mu(ds).$$

We now formulate our basic assumptions in the stochastic case.

(A1) For every $t \in \mathbb{N}$, the function $u_t : S \to \mathbb{R}$ is increasing, strictly concave and continuous.
(A2) For every $t \in \mathbb{N}$, the function $v_t : S \to \mathbb{R}$ is bounded, increasing and continuous.

Assuming in the sequel that (A) holds we mean that both (A1) and (A2) are satisfied. Similarly, we shall refer to other conditions made below.

For the transition probability functions we accept two alternative sets of conditions (B) or (C).

(B) For every $t \in \mathbb{N}$, the transition probability q_t is weakly continuous on S, that is, if $y_m \to y_0$ in S as $m \to \infty$, then $q_t(\cdot|y_m)$ converges weakly to $q_t(\cdot|y_0)$ for every $t \in \mathbb{N}$. Moreover, for each $y \in S_+$, the probability measure $q_t(\cdot|y)$ is non-atomic, and $q_t(\cdot|0)$ has no atoms in S_+.
(C1) For every $t \in \mathbb{N}$, the transition probability q_t is weakly continuous on S and $q_t(\{0\}|0) = 1$.

(C2) For every $s \in S$ and $t \in \mathbb{N}$, the set $Z_t^s := \{y \in S: \ q_t(\{s\}|y) > 0\}$ is countable.

(C3) The transition probability q_t is stochastically increasing, i.e., if $s' \to Q_t(s'|y) := q_t([0, s']|y)$ is the cumulative distribution function for $q_t(\cdot|y)$, then for all $y_1 < y_2$ and $s' \in S$, we have that $Q_t(s'|y_1) \geq Q_t(s'|y_2)$ for every $t \in \mathbb{N}$.

A typical representation of the transition probabilities satisfying assumption (B) is as follows. Let

$$s_{t+1} = \bar{f}_t(y_t, z_t),$$

where $y_t = s_t - a_t$ is the investment in state s_t, $(z_t)_{t\in\mathbb{N}}$ is a sequence of i.i.d. random "shocks" having a probability distribution π. The functions $\bar{f}_t$ for $t \in \mathbb{N}$ are continuous and for any Borel set C in S and investment $y \in S$

$$q_t(C|y) = \int_S 1_C(\bar{f}_t(y, z))\pi(dz)$$

where 1_C is the indicator function of the set C. We would like to point out three special cases:

(D1) $\bar{f}_t(y_t, z_t) = z_t f_t^1(y_t) + (1 - z_t) f_t^2(y_t)$, where $f_t^1, f_t^2 : S \to S$ are continuous increasing functions such that $f_t^1(y) < f_t^2(y)$ for $y \in S_+$ and $f_t^1(0) = f_t^2(0) = 0$ for all $t \in \mathbb{N}$. In addition, π is a non-atomic probability measure on $[0, 1]$.

(D2) The model with *additive shocks*: $\bar{f}_t(y_t, z_t) = f_t(y_t) + z_t$, where $f_t : S \to S$ is a continuous increasing function for each $t \in \mathbb{N}$. The probability measure π is non-atomic with support included in $[0, +\infty)$.

(D3) The model with *multiplicative shocks*: $\bar{f}_t(y_t, z_t) = f_t(y_t)z_t$, where f_t is as in (D2) and the probability measure π is non-atomic with support included in $[0, +\infty)$.

We conclude with an example of transition probability satisfying conditions (C).

(D4) Let $\{\alpha_t\}_{t\in\mathbb{N}}$ be a set of numbers from the interval $(0, 1)$. Define

$$q_t(\cdot|y) := \alpha_t \delta_{f_t^1(y)}(\cdot) + (1 - \alpha_t)\delta_{f_t^2(y)}(\cdot), \qquad y \in S, \quad t \in \mathbb{N},$$

where f_t^1 and f_t^2 are as in (D1) and $\delta_{f_t^i(y)}(\cdot)$ is the Dirac measure concentrated at the point $f_t^i(y)$. Obviously, this transition probability satisfies (C1) and (C3). In order to see that (C2) also holds, let $s \in S$ and note that $Z_t^s = \{y \in A(s) : f_t^1(y) = s \text{ or } f_t^2(y) = s\}$ consists of at most two elements. In the pure deterministic case (with only one function f_t^1) Z_t^s has at most one point.

3. MPE in non-stationary stochastic bequest games

We consider a special subclass F of functions in Φ defined as follows. A function ϕ belongs to F if and only if it is upper semicontinuous and the function $s \to s - \phi(s)$ is non-decreasing on S. Note that every $\phi \in F$ is continuous from the left and has at most countably many discontinuity points.

Assume that F is endowed with the standard *topology of weak convergence*. Recall that a sequence $(\phi_n)_{n\in\mathbb{N}}$ converges to $\phi \in F$ if and only if $\phi_n(s) \to \phi(s)$ as $n \to \infty$ at any continuity point s of ϕ. Applying Helly's theorem for the functions from F restricted to any bounded interval $[0, m]$ ($m \in \mathbb{N}$) and using the standard diagonal method as in Lemma 1 in [4], one can prove the following auxiliary result.

Lemma 1. *F is a metrizable and compact topological space.*

Let $c_{t+1} \in F$. Define

$$BC_t(c_{t+1})(s) := \arg\max_{a \in A(s)} P_t(a, c_{t+1})(s).$$

We say that $BC_t(c_{t+1})(s)$ is the set of all best reply consumption levels of generation $t \in \mathbb{N}$ in state s, given that the following generation is going to use a consumption strategy c_{t+1}. This set is *non-empty and compact* by Lemmas 3.7 and 3.8 in [2], if conditions (A) and (B) hold, or by Lemma 1 in [3], if assumptions (A) and (C) are satisfied. For any $s \in S$ and $t \in \mathbb{N}$, let us define

$$bc_t(c_{t+1})(s) = \max BC_t(c_{t+1})(s). \tag{2}$$

Lemma 2. *Assume that* (A) *and* (B) *hold or* (A) *and* (C) *are satisfied. Then, for each $t \in \mathbb{N}$, bc_t maps F into itself and is continuous.*

Proof. Under assumptions (A) and (B) the conclusion follows from the proof of Theorem 3.1 in [2]. If, on the other hand, (A) and (C) are assumed, then the continuity of bc_t follows from the proof of Theorem 1 in [3], whereas the fact that bc_t maps F into F is proved in Lemma 8 in [3]. Although these proofs concern bequest games on bounded state space $S = [0, \bar{s}]$ (for some $\bar{s} > 0$), they also work for $S = [0, +\infty)$. □

Define the composition of continuous mappings bc_τ as follows:

$$bc_t^m(c) := bc_t \circ bc_{t+1} \circ \cdots \circ bc_{t+m}(c),$$

where $t,\ m \in \mathbb{N}$, $c \in F$.

Let $bc_t^m(F) := \{bc_t^m(c) : c \in F\}$ and

$$F_t^* := \bigcap_{m \in \mathbb{N}} bc_t^m(F).$$

Our first main result is as follows.

Proposition 1. *If* (A) *and* (B) *hold or* (A) *and* (C) *are satisfied, then F_τ^* is non-empty and compact for each $\tau \in \mathbb{N}$ and*

$$F_t^* = bc_t(F_{t+1}^*) \quad \textit{for all} \quad t \in \mathbb{N}. \tag{3}$$

Proof. By Lemma 2, if $G \subset F$ is compact, then the set $bc_t^m(G)$ is also compact in F. It is easy to see that $bc_t^{m+1}(G) \subset bc_t^m(G)$ for all $t, m \in \mathbb{N}$ and $G \subset F$. Clearly, every set F_t^* is non-empty. Note that $bc_t^m(F) \supset bc_t(F_{t+1}^*)$ for all $m \in \mathbb{N}$. Hence $F_t^* \supset bc_t(F_{t+1}^*)$. Let $c_t \in F_t^*$. Then, $c_t \in bc_t^{m+1}(F)$ for all $m \in \mathbb{N}$. Consequently, for each $m \in \mathbb{N}$, there exists some $g_{t+1}^m \in bc_{t+1}^m(F) \subset F$ such that $c_t = bc_t(g_{t+1}^m)$. By compactness of F, without loss of generality, we can assume that $g_{t+1}^m \to g_{t+1} \in F$ as $m \to \infty$. Since the sequence of sets $(bc_{t+1}^m(F))_{m \in \mathbb{N}}$ is decreasing, we conclude that $g_{t+1} \in F_{t+1}^*$. By the continuity of the mapping bc_t, it follows that $c_t = \lim_{m \to \infty} bc_t(g_{t+1}^m) = bc_t(g_{t+1})$. Hence, $c_t \in bc_t(F_{t+1}^*)$. Thus, we have shown that (3) holds. □

Corollary 1. *Under assumptions of Proposition 1, the game with utility (1) has an MPE $(c_t^*)_{t \in \mathbb{N}}$ with $c_t^* \in F$ for each $t \in \mathbb{N}$.*

Proof. Choose any $c_1^* \in F_1^*$. By (3) there exists $c_2^* \in F_2^*$ such that $c_1^* = bc_1(c_2^*)$ and $c_3^* \in F_3^*$ such that $c_2^* = bc_2(c_3^*)$ and so on. In this way we obtain a sequence $(c_t^*)_{t \in \mathbb{N}}$, which is a Markov perfect equilibrium. □

Remark 1. We can weaken our assumption, allowing each function v_t to be unbounded, but then an additional condition on each q_t is required. For example, we can impose that there exists a continuous increasing function $y \to I_t(y)$ such that $q_t([0, I_t(y)]|y) = 1$, see [8] for a similar assumption in standard stochastic games of resource extraction.

4. MPE in non-stationary deterministic bequest games

It happens that within the deterministic framework we are allowed to consider even more general *non-additive utility functions*. We assume that the generation $t+1$'s inheritance or capital is $s_{t+1} = f_t(y_t)$, where $f_t : S \to S$ is a *production function* and y_t is an investment of generation t. Generation t's satisfaction depends on its own consumption $a_t \in A(s_t)$ and on the next generation's consumption $a_{t+1} \in A(s_{t+1})$ and is equal to $U_t(a_t, a_{t+1})$, where $U_t : S \times S \to \mathbb{R}$ is a *utility function*.

Definition 2. A *Markov Perfect Equilibrium* (MPE) in the model with the deterministic production function is a sequence of strategies $(c_t^*)_{t\in\mathbb{N}}$ such that $c_t^* \in \Phi$ for each $t \in \mathbb{N}$ and

$$\sup_{a\in A(s)} U_t(a, c_{t+1}^*(f_t(s-a))) = U_t(c_t^*(s), c_{t+1}^*(f_t(s - c_t^*(s))))$$

for every $s \in S$ and $t \in \mathbb{N}$.

Let $b > 0$ and $\delta : [b, \infty) \to R$ be a fixed function. Following Milgrom and Shannon [15], we say that δ has the *strict single crossing property* on $[b, \infty)$, when the following holds: if there exists some $x \geq b$ such that $\delta(x) \geq 0$, then for each $x' > x$, we have $\delta(x') > 0$. Note that both functions $x \to -\frac{1}{x}$ and $x \to \frac{\ln x}{x}$ have the strict single crossing property on $[b, \infty)$ with $b > 0$ and the latter is not increasing on its domain.

We accept the following assumptions.

(E1) For every $t \in \mathbb{N}$, the function U_t is bounded from below, continuous on $S \times S$ and increasing in each variable.

(E2) For any $y_2 > y_1$ in S, $h > 0$ and $t \in \mathbb{N}$, the function $\Delta_h U_t(x) := U_t(x, y_2) - U_t(x+h, y_1)$ has the strict single crossing property on $[b, +\infty)$ for each $b > 0$.

(E3) For every $t \in \mathbb{N}$, the function $f_t : S \to S$ is continuous and increasing with $f_t(0) = 0$.

Let $c_{t+1} \in F$. We define

$$BC_t(c_{t+1})(s) := \arg\max_{a\in A(s)} U_t(a, c_{t+1}(f_t(s-a)))$$

and

$$bc_t(c_{t+1})(s) := \max BC_t(c_{t+1})(s).$$

Under assumptions (E) the set $BC_t(c_{t+1})(s)$ is non-empty and compact for each $t \in \mathbb{N}$ and $s \in S$. This fact is a consequence of Lemma 4 in [4]. Therefore, the mapping bc_t is well-defined. The sets F_t^* and equation (3) can also be considered in the present non-additive utility case.

Lemma 3. *Assume that* conditions (E) *hold. Then, for each $t \in \mathbb{N}$, bc_t maps F into itself and is continuous.*

Proof. Proposition 1 and Lemma 6 in [4] imply that bc_t maps F into itself. The continuity of bc_t is shown in the proof of Theorem 1 in [4]. □

The proofs of the following result is similar to that of Proposition 1 and is based on Lemma 3.

Proposition 2. *Assume that conditions* (E) *hold. Then, F^*_τ is non-empty and compact for each $\tau \in \mathbb{N}$ and (3) is satisfied.*

Corollary 2. *Under assumptions* (E) *the game with utilities $(U_t)_{t\in\mathbb{N}}$ has an MPE $(c^*_t)_{t\in\mathbb{N}}$ with $c^*_t \in F$ for each $t \in \mathbb{N}$.*

5. Concluding comments

Below we give some bibliographical notes and remarks on the results provided in this note.

Remark 2. The "set-valued recursive equation" (3) constitutes certain *characterization* of the set of MPE in both deterministic and stochastic cases. If the model is *stationary*, i.e., $u_t = u$, $v_t = v$ and $q_t = q$ for all $t \in \mathbb{N}$, then we can define $bc := bc_t$ choosing any $t \in \mathbb{N}$. Hence, we have $F^*_t =: F^*$ for each $t \in \mathbb{N}$. Moreover, $F^* = bc(F^*)$, i.e., the set F^* is invariant under the best response mapping. Observe that this property of F^* is not sufficient to get a stationary MPE (where $c^*_t = c^*_{t+1}$ for all $t \in \mathbb{N}$) in the stationary model. We can only say that in equilibrium $(c^*_t)_{t\in\mathbb{N}}$ every $c^*_t \in F^*$. To obtain stationarity of equilibrium, an additional non-trivial tool as the Schauder–Tychonoff fixed point theorem must be applied. But then equation $bc(F^*) = F^*$ is not needed. One can apply the fixed point argument to the mapping $bc : F \to F$, see [2–4] for details. Here, the derivation of (3) is simpler. However, one should note that Lemmas 2 and 3 on the continuity of bc play a crucial role in the non-stationary case.

Remark 3. The importance of the class F of strategies in studying *deterministic* intergenerational games was already noticed by Bernheim and Ray [7] and Leininger [12]. Bernheim and Ray [7] and Leininger [12] also studied non-stationary MPE in deterministic games with compact state spaces. However, their methods were different in many respects. Instead of working directly with strategies from the set F, Bernheim and Ray [7] worked with some "filled graphs" of the functions from F. As the authors themselves wrote on page 12: "the proofs are rather intricate." We agree with this opinion and note that their proof is long and it is not clear how to extend the results to the unbounded state space $S = [0, \infty)$. Leininger [12], on the other hand, used a "levelling" method that associates with any function $c \in F$ a uniformly continuous function on the compact state space S. This idea was applied to obtain an MPE in the stationary model as well as a stationary MPE. The latter case additionally requires an application of the Schauder fixed point theorem for continuous transformations of compact convex sets in the Banach space of continuous functions on the compact set S endowed with the supremum norm. We would like to emphasize that the method used in [4] and in this note is more direct. It is based on the continuity result for bc_t (Lemma 3). Such continuity result was reported neither in [7] nor [12].

Remark 4. The existence of (non-stationary) MPE for deterministic bequest games was proved in [12] (for a compact state space) and in [4] (for an unbounded state space). However, the above-mentioned papers do not include any characterization of the set of Markov perfect equilibria. The idea was to consider, for each $n \in \mathbb{N}$, a game G_n with active n generations, that is, every generation t with $t > n$ chooses the zero consumption function. An MPE in G_n, say $f^n = (f_t)_{t\in\mathbb{N}}$, is constructed by backward induction method. Clearly, $f_t = 0$ for $t > n$. An MPE in the original bequest game is then obtained as a limit of some subsequence chosen from $(f^n)_{n\in\mathbb{N}}$ by standard diagonal method. In this note, the sets $bc^m_t(F)$ are also constructed by backward induction. In this way, we obtain a nested family of compact sets having a non-empty intersection. The idea of deriving an MPE from (3) is new, but it resembles to some extent the technique utilized in [1,9,13]. For example, Harris [9] and Mertens and Parthasarathy [13] applied akin ideas to prove the existence of subgame perfect equilibria in games of perfect information and discounted stochastic games, respectively. Furthermore, Abreu et al. [1] used a similar tool to establish the existence of a sequential equilibrium in a class of repeated games with imperfect monitoring. Finally, Balbus and Woźny [5] recently provided a characterization of MPE in stochastic games of resource extraction with quasi-hyperbolic discounting and special transition probability functions.

Remark 5. It is worth mentioning that our assumptions (E1) and (E2) embrace the ones used by Bernheim and Ray [7]. More precisely, Bernheim and Ray [7] (in order to prove their results, see Theorem 4.2) assume that U_t satisfies (E1), U_t is strictly concave in its first argument and

(ID) U_t has increasing differences, i.e., for any $y_2 > y_1$ in S the function $\Delta U_t(x) := U_t(x, y_2) - U_t(x, y_1)$ is non-decreasing.

Hence, in this paper not only do we replace their increasing differences assumption on U_t by our weaker condition (E1), but most importantly our method allows to study bequest games with *unbounded payoffs* and *unbounded state spaces*. The reader is referred to Examples 1-3 in [4], which illustrate that our conditions are indeed more general than those used in [7].

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