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On uniqueness of time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty [☆]

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Abstract

We give a set of sufficient conditions for uniqueness of a time-consistent stationary Markov consumption policy for a quasi-hyperbolic household under uncertainty. To the best of our knowledge, this uniqueness result is the first presented in the literature for general settings, i.e. under standard assumptions on preferences, as well as some new condition on a transition probability. This paper advocates a "generalized Bellman equation" method to overcome some predicaments of the known methods and also extends our recent existence result. Our method also works for returns unbounded from above. We provide a few natural extensions of optimal policy uniqueness: convergent and accurate computational algorithm, monotone comparative statics result and generalized Euler equation.

JEL classification: C61; C73

Keywords: Time consistency; Markov equilibrium; Uniqueness; Generalized Bellman equation

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1. Introduction

The problem of dynamic inconsistency in sequential decision models was introduced in the seminal paper of Strotz (1956), further developed in the work of Phelps and Pollak (1968) and Peleg and Yaari (1973), and has played an increasingly important role in many fields in economics (see Sorger, 2004; Nakajima, 2012; Bernheim et al., 2015; Drugeon and Wigniolle, 2016; Jackson and Yariv, 2014, 2015; Chatterjee and Eyigungor, 2016; Echenique et al., 2016 for some recent contributions). The classical toolkit for analyzing "time" consistency problems introduced in Strotz (1956) emphasized the language of recursive decision theory. In this approach, for sequential optimization problems with dynamically inconsistent objectives, one constructs dynamically consistent plans by imposing additional constraints on the agent's decision problem (e.g., also see Kydland and Prescott, 1980). It is well-known that these constraints can be problematic to formulate. As has been observed by many researchers in subsequent work (e.g., Peleg and Yaari, 1973 and Bernheim and Ray, 1986), even the existence of such optimal dynamically consistent plans in the class of Markovian solutions cannot be guaranteed, let alone the question of their uniqueness. Further, when such dynamically consistent optimal plans are known to exist, they can be difficult to characterize and/or compute (see Caplin and Leahy, 2006).

As a way of circumventing these serious problems, Peleg and Yaari (1973) proposed a dynamic game interpretation of the time-consistency problem. In this view, one envisions the decisionmaker as playing a dynamic game between one's current self, and each of one's future "selves", with the solution concept for consistent plans being a subgame-perfect Nash equilibrium. But such an approach does not always simplify the analysis, as even when this approach to dynamic consistency is followed, the question of the existence of subgame-perfect Nash equilibrium is still not a trivial matter, nor is the question of sufficient conditions for the equilibrium uniqueness in pure Markovian strategies (see Bernheim and Ray, 1986 and Leininger, 1986).[2] Finally, it bears mentioning the recent important work of Maliar and Maliar (2006, 2016), where it is shown that providing sharp numerical algorithms that compute equilibria in the quasi-hyperbolic discounting problem even when *unique* Markovian equilibrium exists can be a difficult problem.[3]

In this paper, we seek conditions under which there exists a globally stable monotone iterative numerical algorithm that: (i) characterizes the existence *and* uniqueness of pure strategy Markovian equilibrium, (ii) provides an explicit and accurate method for computing pure strategy Markovian equilibrium, and (iii) facilitates the characterization of monotone comparative statics in the deep parameters of the model. From the perspective of the existence question, our paper is closely related to the important papers of Bernheim and Ray (1986) or Harris and Laibson (2001), where the authors add noise of invariant support of the game to develop conditions that guarantee the existence of a time-consistent policy in a class of Markovian equilibria that are locally of bounded variation and Lipschitz for sufficiently small hyperbolic discount factor. We should also mention, there is a critical difference between approaches taken in this literature (e.g., Harris and Laibson, 2001, and many papers subsequent), and many of those advocated in the present paper: our methods do *not* rely on so-called "generalized Euler equations". Rather, in our paper, we propose a "generalized Bellman equation" approach, where a new "value itera-

[2] The works of Kocherlakota (1996) and Maskin and Tirole (2001) provide an extensive set of motivations for why one might be interested in concentrating on equilibria in Markovian strategies as opposed to subgame perfect equilibria.

[3] We should mention, uniqueness in these models is within a "class" of Markovian equilibrium, where closed-form solutions are available.

tion" method is constructed that provides a globally stable "generalized value iteration" method for computing equilibrium values/policies. The approach we advocate can provide an important new link between the methods in the existing literature based on generalized Euler equation methods (which are a type of generalized policy iteration methods), and new methods based on our generalized Bellman approach. Indeed, under some conditions, we show our generalized value iteration methods imply the existence of generalized Euler equation methods. But the latter approach requires stronger conditions than necessary for computing unique time consistent Markovian equilibrium. Further, our new approach allows us to link the stochastic games approach studied in Harris and Laibson (2001), with a recursive optimization/value function methods suggested by Strotz (1956), and further developed by Caplin and Leahy (2006). In particular, we give sufficient conditions on primitives where generalized Euler equation methods are *globally valid.*

We should mention the recent contribution in Chatterjee and Eyigungor (2016), where the authors propose an interesting method to show existence of a *continuous* randomized Markov perfect equilibria in a quasi-hyperbolic discounting model with a strictly positive lower bound on wealth. Specifically, they show that once consumers are allowed to randomize their investment strategies (keeping the expected investment constant) they will endogenously choose a strategy that concavifies the expected continuation value function. Our approach is similar in spirit to their approach, but attacks the problem from the vantage point of a stochastic game (as opposed to using a lotteries approach). This is an important difference between these two approaches. In particular, in our paper, we place conditions on the *primitives* of a stochastic game that in essence "concavify" the continuation expected utility exogenously in a similar manner to the lotteries approach in Chatterjee and Eyigungor (2016). Aside from not requiring one to resort to lotteries, our approach has the additional benefit relative to Chatterjee and Eyigungor (2016) of allowing us to state simple sufficient conditions on primitives for obtaining unique equilibrium in pure *Lipschitzian* Markov strategies.

More specifically, under standard assumptions on preferences, and some condition on a transition probability that has been applied extensively in the existing literature on stochastic games, we are able to develop a monotone value iteration approach to show existence and uniqueness of equilibrium policy. Further, we are able to provide sharp characterizations of their Lipschitzian structure, as well as their monotonicity properties. Finally, and equally as important, as we obtain sufficient conditions for the uniqueness of Markovian equilibrium policy on a *minimal state space* and work for returns that are bounded or unbounded above. Therefore, our new methods nicely complement those of Chatterjee and Eyigungor (2016) on the existence of continuous Markovian equilibria in such time consistency problems.

We are also able to construct a simple successive approximation scheme for computing unique pure strategy Markovian equilibrium values in the appropriate norm, as well as conduct monotone comparative statics with the model parameters. These comparative statics and approximation results are important for applied research in the field. For example, in Sorger (2004), he proposes settings under which any twice continuously differentiable function can be supported as a policy of a time consistent hyperbolic consumer. This result can be subsequently linked to a Gong et al. (2007), where it is shown that a hyperbolic discounting is not observationally equivalent to exponential discounting. However, the two models have *radically different* comparative statics. Hence, our approach allows us characterize an exact answer to this question, and provide theoretical monotone comparative statics to clarify empirical questions that are asked by applied researchers. Finally, being motivated by payoff specifications in applied work, as an important technical matter, we are able in our setting to dispense with the requirement of bounded returns above. This allows our results to hold for many utility specifications common in the applied literature on quasi-hyperbolic discounting.

2. Main results

The environment we study is standard in the literature on quasi-hyperbolic discounting. We have an individual decision maker who views herself as a sequence of "selves" indexed in discrete time $t \in \{0, 1, \ldots\}$. In any period t, given output or asset level (i.e. state) $s_t \in S$ (where $S = [0, \infty)$), "self t" chooses a consumption $c_t \in [0, s_t]$, and then she leaves $i_t := s_t - c_t$ as investment for her future "selves". As in effect we are ruling out borrowing, we interpret the asset held by any self t as being a productive one (and hence, we shall typically refer to this asset as "capital"). Together with current output level s_t, these choices (consumption and hence investment levels) in the current period t determine a transition probability $Q(ds_{t+1}|s_t - c_t, s_t)$ for next period's output or productive asset holding (or simply state s_{t+1}).

Self t preferences are represented by a infinite horizon utility function given by:

$$u(c_t) + \beta E_t \sum_{i=t+1}^{\infty} \delta^{i-t} u(c_i),$$

where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, u is a instantaneous utility function and expectations E_t are taken with respect to a realization of a random variable s_i drawn each period from a transition distribution Q, and will be well-defined by the Ionescu–Tulcea theorem.

2.1. Generalized Bellman operator

Under some natural continuity assumptions on u and Q (to be specified later), we can define a pure strategy Markovian equilibrium to be an $h \in \mathcal{H}$, where $\mathcal{H} = \{h : S \to S | 0 \leq h(s) \leq s, h \text{ is Borel measurable}\}$, where h satisfies the following functional equation:

$$h(s) \in \arg\max_{c \in [0,s]} u(c) + \beta\delta \int_S V_h(s') Q(ds'|s - c, s), \tag{1}$$

where $V_h : S \to \mathbb{R}$ is a continuation value function for the household of "future" selves following a stationary policy h from tomorrow on. The value function in such a pure strategy Markovian equilibrium for the future selves, therefore, must solve the following additional functional equation in the continuation given as follows:

$$V_h(s) = u(h(s)) + \delta \int_S V_h(s') Q(ds'|s - h(s), s).$$

In our case such a pure strategy Markovian equilibrium is also time-consistent policy for quasi-hyperbolic consumer.

Therefore, if we define the value function for the self t to be:

$$W_h(s) := u(h(s)) + \beta\delta \int_S V_h(s') Q(ds'|s - h(s), s),$$

we obtain the fundamental relation we study in this paper:

$$V_h(s) = \frac{1}{\beta} W_h(s) - \frac{1-\beta}{\beta} u(h(s)). \tag{2}$$

Equation (2) is our *generalized Bellman equation*. Solutions to this functional equation characterize a Markovian value function that solves our original maximization problem, where the element $\frac{1-\beta}{\beta} u(h(s))$ is the adjustment that must be made to the standard Bellman operator to account for changing preferences. That is, for $\beta = 1$, equation (2) reduces to the standard Bellman equation. Based on equation (2), we can define an operator whose fixed points, say V^*, correspond to values for some pure strategy Markovian equilibrium policy. From there, we can recover the set of (pure strategy) Markovian equilibrium policy functions.

2.2. *Assumptions*

Unlike recent work in the literature, we allow for period returns to be unbounded above (but bounded below). To do this, we need to introduce some definitions. Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact subsets of S that are increasing under the set inclusion partial order such that each of K_j contains 0, and let the state space be $S = \bigcup_{j=1}^{\infty} K_j$. For $V : S \mapsto \mathbb{R}$, V bounded on each K_j, $j \in \mathbb{N}$, define the collection of seminorms (see Matkowski and Nowak, 2011):

$$||V||_j := \sup_{s\in K_j} |V(s)|.$$

Put $m_j := \frac{||u||_j}{1-\beta}$, and define:

$$||V|| := \sum_{j=1}^{\infty} \frac{||V||_j}{m_j} \beta^j,$$

with the convention $||V|| = \infty$, if the series on the right hand side above tends to ∞. By $M(S)$, we denote a set of real-valued, positive, and Borel measurable functions on S. Consider a vector space

$$\mathcal{V} := \left\{ V \in M(S) : V(0) = 0,\ \text{and for all } j \in \mathbb{N}, ||V||_j < \infty,\ \text{and } ||V|| < \infty \right\}$$

and denote

$$\mathcal{V}^m := \{V \in \mathcal{V} : ||V||_j \le m_j \text{ for each } j \in \mathbb{N}\}.$$

We can now state our fundamental assumption on the primitives for our stochastic game:

Assumption 1. Let us assume:

- $u : S \to \mathbb{R}_+$ is continuous, increasing and strictly concave with $u(0) = 0$;
- for any $s, i \in S$ $Q(\cdot|i, s) = p(\cdot|i, s) + (1 - p(S|i, s))\delta_0(\cdot)$, where δ_0 is a delta Dirac measure concentrated at point 0, while $p(\cdot|i, s)$ is some measure such that
  - for each $s \in S \setminus \{0\}$, $i \in [0, s]$ $p(S|i, s) < 1$ and $p(S|0, 0) = 0$;
  - for each $V \in \mathcal{V}^m$, the function

    $$(i, s) \mapsto \int_S V(s') p(ds'|i, s)$$

    is continuous with (i, s) but increasing and concave with i;

  – for each $j \in \mathbb{N}$, $p(K_{j+1}|i,s) = p(S|i,s)$ if $s \in K_j$ and $i \in [0,s]$;
- the sequence $(m_j)_{j\in\mathbb{N}}$ satisfies

$$\delta \sup_{j\in\mathbb{N}} \left\{ \frac{m_{j+1}}{m_j} \right\} \leq \beta.$$

We make a few remarks.

First, our assumptions on preferences are completely standard. That is, here, we only impose the strict concavity of a period utility functions to restrict attention to single valued best replies in the equation (1). This allows us to study a single-valued operator whose fixed points generate corresponding equilibrium values and corresponding time consistent policies. It bears mentioning, a careful reading of the proof of our main existence theorem below (Theorem 1) indicates this assumption can be *weakened* if the existence (but not uniqueness) of pure strategy Markovian equilibrium is all that one seeks, as in the case of multi-valued best reply maps, one can simply work with the increasing selections/ascending correspondence from a best response map.[4]

Second, our assumption on the transition probability Q requires a few remarks. First, although this is a powerful technical assumption, the conditions is satisfied in many applications (e.g., see the discussion in Chassang (2010) for a particular example of this exact structure). Additionally, observe Q is a mixture of measures p and δ_0. That is, one gets a draw from p or (with measure $1-p$) ends in 0, which again by our assumption is an absorbing state. The most restrictive part of our assumption, however, is a requirement that for any positive and integrable function v, the mapping $v\colon i \mapsto \int_S V(s')p(ds'|i,s)$ is increasing and concave with i. This means that the higher the investment, the larger the measure $p(\cdot|i,s)$ for each measurable set. Indeed, as we assume positive returns (i.e., $u(\cdot) \geq 0$), our assumptions above assure that the expected continuation value is monotone and concave in its arguments. For this to be meaningful, we require that $p(\cdot) < 1$; hence, an absorbing state 0 must be always attainable with nonzero probability.

Although this stochastic structure is restrictive, it is common in the stochastic games literature. For example, a *stronger* version of this assumption was introduced by Amir (1996), used extensively in a series of papers by Nowak (see Nowak, 2006; Balbus and Nowak, 2008 and references within), as well as in the context of stochastic games with strategic complementarities with public information in Balbus et al. (2014). We refer the reader to our related paper (see Balbus et al., 2015) for a detailed discussion of the nature of these assumptions.

Third, we can provide many simple examples of where our assumptions on the transition probability Q are satisfied. A typical example would have p be given as follows: $p(\cdot|i,s) = \sum_{j=1}^{J} g_j(i,s)\eta_j(\cdot|s)$, where $\eta_j(\cdot|s)$ are measures on $\mathcal{S}$ (set of subsets of S), where each $g_j : S \times S \to [0,1]$ is an element of the set of continuous functions, that are additionally increasing and concave for each i, with $\sum_{j=1}^{J} g_j(\cdot) \leq 1$ and $g_j(0,0) = 0$. To see in this setting our assumptions on Q are satisfied, note we have:

$$Q(\cdot|i,s) = \sum_{j=1}^{J} g_j(i,s)\eta_j(\cdot|s) + (1 - \sum_{j=1}^{J} g_j(i,s)\eta_j(S|s))\delta_0(\cdot).$$

In this case, Q becomes a linear combination of measures $\{\eta_j\}_{j=1}^{J}$ and δ_0, all independent on i, with $g_j(i,s)$ interpreted as a probability of obtaining a draw from η_j, while

[4] E.g., via Topkis' theorem, the best reply correspondence is strong set order ascending with least and greatest increasing selections.

$1-\sum_{j=1}^{J} g_j(i,s)\eta_j(S|s)$ a probability of obtaining a draw from $\delta_0(\cdot)$. When integrated over $v \in \mathcal{V}$, this gives:

$$\begin{aligned}\int_S V(s')Q(ds'|i,s) &= \sum_{j=1}^{J} g_j(i,s) \int_S V(s')\eta_j(ds'{}_{j=1}^{\prime J} g_j(i,s)\eta_j(S|s)) \int_S V(s')\delta_0(ds') \\ &= \sum_{j=1}^{J} g_j(i,s) \int_S V(s')\eta_j(ds'{}_{j=1}^{\prime J} g_j(i,s)\eta_j(S|s))V(0) \\ &= \sum_{j=1}^{J} g_j(i,s) \int_S V(s')\eta_j(ds'|s),\end{aligned}$$

as $V(0)=0$. Clearly, for any positive and integrable V, function

$$i \mapsto \sum_{j=1}^{J} g_j(i,s) \int_S V(s')\eta_j(ds'|s)$$

is increasing and concave (as each $g_j \in G_j$ is increasing and concave). Moreover, 0 is an absorbing state as $g_j(0,0)=0$.

Notice, we generally do not require p to be a probability measure; i.e., there are other examples of p, where it cannot be expressed by a linear combination of stochastic kernels, yet still satisfy all of our assumptions.

Fourth, our assumptions on Q imply that for $v \in \mathcal{V}$, the expected value function remains concave. In this sense, our approach is similar to the randomization/lotteries technique advocated in Chatterjee and Eyigungor (2016). To see the relationship between the nature of this assumption on primitives of the stochastic game, and their endogenous concavification result, observe whenever v is not concave at the neighborhood of zero capital, the optimal endogenous randomization mechanism would require choosing an atom at zero (*exactly* as required by our assumption). The difference is that in our setting, our assumption is of a *global* nature (i.e. satisfied for *any* candidate measurable value function V, rather than the local one as in their paper). Hence, in our paper, the question of pure strategy Markovian equilibrium existence/uniqueness can be attacked directly.

Finally, our assumptions impose the needed structure required to construct the sequence of $(m_j)_{j\in\mathbb{N}}$ that can be used to define our collection of semi-norms, as well as $p(K_{j+1}|i,s) = p(S|i,s)$ if $s \in K_j$, $i \in [0,s]$, each required to prove pure strategy Markovian equilibrium existence and uniqueness, when returns are unbounded from above. A special case of our assumption is, when u is in fact bounded on S. Next, note that the assumptions can be even further weakened for the case of a bounded state space S. That is, in both cases (either u bounded or S bounded), assumptions discussed in this paragraph are not required.

2.3. *Pure strategy Markov equilibrium uniqueness*

We start by noting an important auxiliary result.

Lemma 1. *$\mathcal{V}$ is a Banach space and $||\cdot||$ is its norm.*

Proof. It follows from Remark 1 and Lemma 1 in Matkowski and Nowak (2011). □

It is easy to verify, that $\mathcal{V}^m$ is a closed subset of $\mathcal{V}$, hence by Lemma 1 a complete metric space. Following Rincon-Zapatero and Rodriguez-Palmero (2003, 2009) we define k-local contractions:

Definition 1. Let $k \in \{0, 1\}$. An operator $T : \mathcal{V}^m \mapsto \mathcal{V}^m$ is *k-local contraction* with modulus $\gamma \in (0, 1)$ if for each pair $V_1, V_2 \in \mathcal{V}^m$

$$||T(V_1) - T(V_2)||_j \leq \gamma ||V_1 - V_2||_{j+k}.$$

We now construct an operator $T : \mathcal{V} \mapsto \mathcal{V}$ by:

$$TV(s) = \frac{1}{\beta} AV(s) - \frac{1-\beta}{\beta} u(BV(s)),$$

where the pair of operators A and B defined on space $\mathcal{V}^m$ are given by:

$$AV(s) = \max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S V(s'|s-c, s) Q(ds'|s-c, s) \right\},$$

$$BV(s) = \arg\max_{c \in [0,s]} \left\{ u(c) + \beta\delta \int_S V(s'|s-c, s) Q(ds'|s-c, s) \right\}.$$

Notice, in the above, we have defined the operator B to map between candidates for equilibrium values $\mathcal{V}$ to spaces of pure strategy best replies $\mathcal{H}$. That is, in effect, we have a pair of operator equations we need to solve if we are to construct the set of Markov equilibrium values $V^* \in \mathcal{V}$. Recall also:

$$TV(s) = u(BV(s)) + \delta \int_S V(s') Q(ds'|s - BV(s), s).$$

For each $j \in \mathbb{N}$, let $\mathcal{V}_j$ be a set of all restrictions of $\mathcal{V}$ to K_j. Endow, $\mathcal{V}_j$ with natural componentwise order. Before proceeding, we make a useful observation implied by Assumption 1.

Lemma 2. *Assume 1, then for any $V \in \mathcal{V}$:*

$$\int_S V(s') Q(ds'|i, s) = \int_S V(s') p(ds'|i, s).$$

Proof. Indeed:

$$\begin{aligned}\int_S V(s') Q(ds'|i, s) &= \int_S V(s') p(ds'|i, s) + (1 - p(S|i, s)) \int_S V(s') \delta_0(ds') \\ &= \int_S V(s') p(ds'|i, s) + (1 - p(S|i, s)) V(0) = \int_S V(s') p(ds'|i, s). \quad \square\end{aligned}$$

Lemma 3. *Let $j \in \mathbb{N}$, $s \in K_j$, $V_1, V_2 \in \mathcal{V}_j$, and suppose that $V_1(s') \leq V_2(s')$ for each $s' \in K_{j+1}$. Then, $BV_1(s) \geq BV_2(s)$, and $TV_1(s) \leq TV_2(s)$.*

Proof. To see the monotonicity of B, consider a function $G : [0, s] \times S \times \mathcal{V}_{j+1} \mapsto \mathbb{R}$

$$G(c, s, V) = u(c) + \beta\delta \int_S V(s')p(ds'|s - c, s).$$

We now show that $c \to G(c, s, V)$ is supermodular and has decreasing differences in (c, V). Indeed, for any $V \in \mathcal{V}_j$ the function $G(\cdot, s, V)$ is trivially supermodular on $[0, s] \subset \mathbb{R}$.

Moreover, $(c, V) \to \int_S V(s')p(ds'|s-c, s)$ has decreasing differences. To show this, let $V_2 \geq V_1$ and $c_2 \geq c_1$. By Assumption 1 for each s, $p(\cdot|s - c_1, s) - p(\cdot|s - c_2, s)$ is a signed measure. That is, observe for each Borel set S', $\epsilon \cdot \mathbf{1}_{S'} \in \mathcal{V}^m$ for sufficiently small $\epsilon > 0$, by Assumption 1, we have $\epsilon \cdot (p(S'|s-c_1, s) - p(S'|s-c_2, s)) \geq 0$. Therefore, this difference is a measure. Further, we have

$$\begin{aligned}
0 &\leq \int_S V_1(s')p(ds'|s - c_1, s) - \int_S V_1(s')p(ds'|s - c_2, s), \\
&= \int_S V_1(s')[p(ds'|s - c_1, s) - p(ds'|s - c_2, s)], \\
&\leq \int_S V_2(s')[p(ds'|s - c_1, s) - p(ds'|s - c_2, s)].
\end{aligned}$$

Therefore, the function $(c, V) \to G(c, s, V)$ has decreasing differences on $[0, s] \times \mathcal{V}_{j+1}$. Since $[0, s]$ is a lattice,[5] and $\mathcal{V}_{j+1}$ is a poset, we obtain by Topkis (1978) theorem, the (unique) best reply $BR(V)(s) = \arg\max_{c \in [0,s]} G(c, s, V)$ is decreasing on $\mathcal{V}_{j+1}$. Since A is increasing, and B decreasing, by definition of T, we have T is increasing. □

The following lemma is straightforward to prove.

Lemma 4. *For each $j \in \mathbb{N}$, $V \in \mathcal{V}$, $s \in K_j$ and constant $k \in \mathbb{N}$, $B(V + k)(s) = BV(s)$, and $A(V + k)(s) = AV(s) + \beta\delta k$. As a result, $T(V + k)(s) = TV(s) + \delta k$.*

Lemma 5. *T maps $\mathcal{V}^m$ into itself.*

Proof. Let $V \in \mathcal{V}^m$, $j \in \mathbb{N}$ and $s \in K_j$ be given. Observe $BV(s) \in K_j$. Then, by the definition of T, we have

$$\begin{aligned}
TV(s) &\leq (1 - \beta)m_j + \delta \int_{K_j} V(s')Q(ds'|s - BV(s), s) \leq (1 - \beta)m_j + \delta m_{j+1} && (3) \\
&\leq (1 - \beta)m_j + \beta m_j = m_j. && (4)
\end{aligned}$$

Here, (3) follows from our assumption on the sequence of $(m_j)_{j\in\mathbb{N}}$. Since j and s were fixed arbitrarily, (4) implies that $TV(\cdot) \in \mathcal{V}^m$. □

Lemma 6. *$T : \mathcal{V}^m \mapsto \mathcal{V}^m$ is 1-local contraction with modulus δ.*

[5] Recall, a poset X is a lattice, if for any $x, x' \in X$ we have $\sup\{x, x'\} \in X$ and $\inf\{x, x'\} \in X$.

Proof. Let $V_1, V_2 \in \mathcal{V}^m$, $j \in \mathbb{N}$, $s \in K_j$, and put $k_0 := ||V_1 - V_2||_{j+1}$. Then, by Assumption 1, $Q(K_{j+1}|s - BV_i(s), s) = 1$. By Lemma 4, we have

$$T(V_i + k_0)(s) = TV_i(s) + \delta k_0.$$

Further, by Lemma 3,

$$TV_2(s) - \delta k_0 = T(V_2 - k_0)(s) \le TV_1(s) \le T(V_2 + k_0)(s) = TV_2(s) + \delta k_0.$$

Hence, $|TV_1(s) - TV_2(s)| \le \delta k_0$. The proof is now complete as $s \in K_j$ is chosen arbitrary. □

For any fixed point V^* of the operator T, this value function corresponds to a pure, stationary Markov equilibrium policy $h^* = BV^* \in \mathcal{H}$. Equip the space of pure strategies $\mathcal{H}$ with the usual pointwise partial order. In this case, we obtain our main result:

Theorem 1 *(Uniqueness of pure strategy Markovian equilibrium). Let Assumption 1 hold. Then, there is a unique value $V^* \in \mathcal{V}^m$ and corresponding unique pure strategy Markovian equilibrium $h^* \in \mathcal{H}$. Moreover, for any $V \in \mathcal{V}^m$, we have*

$$\lim_{t \to \infty} ||T^t V - V^*|| = 0. \tag{5}$$

Proof. Observe from Lemma 1, $(\mathcal{V}, ||\cdot||)$ is a Banach space; hence $(\mathcal{V}^m, ||\cdot||)$ is complete metric space. Furthermore, by Lemma 5, T maps $\mathcal{V}^m$ into itself, and by Lemma 6, T is 1-local contraction with modulus δ. Therefore, by Theorem of Rincon-Zapatero and Rodriguez-Palmero (2003, 2009) or Matkowski and Nowak (2011), T is a contraction with respect to the metric space $(\mathcal{V}^m, ||\cdot||)$. From standard Banach Contraction Principle, there is unique fixed point of T, $V^* \in \mathcal{V}^m$, and (5) holds. □

Theorem 1 is the central result of our paper. It is important for many reasons.

First, it guarantees existence of pure strategy Markov equilibrium value V^* and policy h^*. Second, it asserts that such an equilibrium value and equilibrium policy is unique, where the uniqueness result holds within a *large* class of functions (i.e., unbounded (from above) or bounded measurable value functions). In turn, this implies that sequences generated by operator T are converging to V^* in the appropriate norm topology.

Such a strong characterization of equilibrium policies is obtained due to two central assumptions: (a) concentrating on Markovian policies and (b) the mixing assumption imposed on Q. Without these assumptions, our results would be substantially weaker. That is, the operator T is a Bellman type operator and expresses the time-consistency problem recursively for Markovian policies. However, generally if Assumption 1 is not satisfied, the mapping T does not have the useful properties of similar Bellman-type operators applied in the study of optimal economies.[6] Finally, although under Assumption 1 T is a contraction, the useful properties concerning equilibrium h^* characterization do not follow from standard arguments used in Stokey et al. (1989). For this reason, we present the result further characterizing the equilibrium policy functions.

[6] It suffices to change δ-Dirac measure with some other nontrivial one in Assumption 1 and equilibrium uniqueness results would not hold. In such a case one could show Markov-equilibrium existence using topological arguments but with no hope of uniqueness. Also equilibrium computation would become substantially complicated (see Maliar and Maliar, 2006, 2016).

Theorem 2 *(Monotonicity of pure strategy Markovian equilibrium policies). Assume 1, and consider a pure strategy Markovian equilibrium h^*. If $p(\cdot|i,s)$ is constant with s, for any i, then h^* is increasing and Lipschitz with modulus* 1.

Proof. Let $h^* = BV^*$ for $V^* = TV^*$. Consider the function

$$G(c,s,V^*) = u(c) + \beta\delta \int_S V^*(s')p(ds'|s-c).$$

First, note G is supermodular in c on a lattice $[0,s]$, and the feasible action set $[0,s]$ is increasing in the Veinott's strong set order.[7] Moreover, by concavity of $i \to \int_S V^*(s')p(ds'|i)$, G has increasing differences with (c,s). Indeed, for any $c_2 \geq c_1$ and $s_2 \geq s_1$ we have:

$$\begin{aligned} 0 &\leq \int_S V^*(s')p(ds'|s_2-c_1) - \int_S V^*(s')p(ds'|s_2-c_2) \\ &\leq \int_S V^*(s')p(ds'|s_1-c_1) - \int_S V^*(s')p(ds'|s_1-c_2), \end{aligned}$$

where in the second inequality follows from concavity (i.e., concave functions have differences that decrease). Next, by Topkis (1978) theorem argument maximizing h^* is increasing with s on S.

Similarly, recalling that i denotes investment, we also can rewrite this problem as:

$$H(i,s,V^*) = u(s-i) + \beta\delta \int_S V^*(s')p(ds'|i),$$

where H is supermodular with the choice variable i on a lattice $[0,s]$, and the set $[0,s]$ is increasing in Veinott's strong set order. Again, by concavity of u, we conclude that H has increasing differences with (i,s). Therefore, by Topkis (1978) theorem, the optimal solution i^* is increasing with s on S.

Clearly $i^*(s) = s - h^*(s)$. Finally as both h^* and i^* are increasing on S, we conclude h^* and i^* are Lipschitz with modulus 1. □

Notice the result in the above theorem is very important, as it extends the result reported in Harris and Laibson (2001) on Lipschitz continuity of equilibrium to a broader scope of quasi-hyperbolic discount factors. It also provides strong structural characterization of pure strategy Markovian equilibrium policies.

2.4. Monotone comparative statics

We next turn to the question of the existence and computation of monotone comparative statics. Aside from the obvious interest is such a question, this is an especially important consideration given the indeterminacy result in Maliar and Maliar (2006); Gong et al. (2007). It is also important, if one wants to study the mode from an econometric point of view.

[7] We say poset X_2 is greater than poset X_1 in the Veinott's strong set order, whenever for any $x_1 \in X_1$ and $x_2 \in X_2$ we have $\sup\{x_1,x_2\} \in X_2$ and $\inf\{x_1,x_2\} \in X_1$.

Along these lines, consider a parameterized version of our optimization problem in the previous section of the paper. For a partially ordered set Θ, with $\theta \in \Theta$ a typical element, define the unique pure strategy Markovian equilibrium as h^*_θ. We make the following assumption.

Assumption 2. Let us assume:

- u does not depend on θ and obeys Assumption 1;
- for any $s, i \in S$ and $\theta \in \Theta$ let $Q(\cdot|i, s, \theta) = p(\cdot|i, s, \theta) + (1 - p(S|i, s, \theta))\delta_0(\cdot)$, where for each θ $p(\cdot|i, s, \theta)$ obeys Assumption 1;
- for each $V \in \mathcal{V}$, we have $(i, \theta) \to \int_S V(s')p(ds'|i, s, \theta)$ has decreasing differences with (i, θ) and $\theta \to \int_S V(s')p(ds'|i, s, \theta)$ is decreasing on Θ.

Lemma 7. *Let $\phi : S \times \Theta \mapsto \mathbb{R}$ be a function such that $\phi(\cdot, \theta) \in \mathcal{V}$ for each $\theta \in \Theta$, and $\phi(s, \cdot)$ is decreasing for each $s \in S$. Then $\theta \mapsto T_\theta(\phi(\cdot, \theta))(s)$ is a decreasing function.*

Proof. It is easy to see for all $V \in \mathcal{V}$, the mapping $\theta \in \Theta \mapsto A_\theta(V)$ is a decreasing function. This follows immediately from Assumption 2. We now show that $B_\theta(V)$ is increasing in θ. For each $s \in S$, define

$$G(c, V, \theta) := u(c) + \beta\delta \int_S V(s')p(ds'|s - c, s, \theta).$$

Suppose that $c_1 < c_2 \le s$. By Assumption 2, we have:

$$\begin{aligned} &G(c_2, V, \theta) - G(c_1, V, \theta) \\ &\quad := u(c_2) - u(c_1) + \beta\delta \int_S V(s')p(ds'|s - c_1, s, \theta) - \beta\delta \int_S V(s')p(ds'|s - c_2, s, \theta) \end{aligned}$$

increasing in θ. Then, by Topkis (1978) theorem, $BV_\theta(V)$ is increasing in θ. Further, by Assumption 2 and Lemma 3, $V \in \mathcal{V} \mapsto B_\theta(V)$ is decreasing. As a result, $B_\theta(\phi(\cdot, \theta))$ is increasing in θ. Furthermore, $\theta \mapsto T_\theta(\phi(\cdot, \theta))(s)$ is decreasing function for any $s \in S$. $\square$

With Assumption 2 in place, we can now prove our main result on monotone comparative statics for extremal equilibrium policies.

Theorem 3 *(Monotone comparative statics). Let Assumption 2 be satisfied. Then, the equilibrium mapping $\theta \to h^*_\theta$ is increasing.*

Proof. Observe by Theorem 1, we have

$$V^*_\theta(s) = \sup_n T^n_\theta(\mathbf{0})(s) = \lim_{n\to\infty} T^n_\theta(\mathbf{0})(s)$$

where $\mathbf{0}$ is a zero function. By Lemma 7, $T_\theta(\mathbf{0})(\cdot) \in \mathcal{V}$ and is decreasing in θ. Consequently, all $T^n_\theta(\mathbf{0})(s)$ satisfy all conditions of Lemma 7; hence $V^*_\theta(s)$ decreases in θ. To finish the proof, observe that $h_\theta(\cdot) = B_\theta(V^*_\theta)(\cdot)$, and hence by Lemma 7, h_θ is increasing in θ. $\square$

2.5. Existence of a generalized Euler equation

Since Harris and Laibson (2001), many researchers have applied the "generalized Euler equation" approach to solving dynamic/stochastic games. In this approach, one essentially assumes a smooth Markovian equilibrium exists, and then develops numerical methods for computing it. The problem is in general, there is not reason to believe such smooth equilibria exist. We now provide sufficient conditions for the existence of a unique *differentiable* pure strategy Markovian equilibrium, and state the version of the generalized Euler equation that characterizes it.

For $V \in \mathcal{V}$, let $F_V(i) := \beta\delta \int_S V(s')Q(ds'|i)$, where $Q(\cdot|i)$ denotes transition $Q(\cdot|i,s)$ that is independent on s.

We first prove a Lemma that shall be used in the sequel:

Lemma 8. *Assume 1. Then, $BV^*(\cdot)$ and $V^*(\cdot)$ are a.e. differentiable.*

Proof. Take any[8] $V \in \mathcal{V}$. Obviously

$$AV(s) := \max_{c\in[0,s]} (u(c) + F_V(s-c)) = \max_{i\in[0,s]} (u(s-i) + F_V(i)). \tag{6}$$

By Assumption 1, u and F_V are concave functions, and u is strictly concave. Similarly, as in the proof of Theorem 2, we obtain that $(i,s) \to u(s-i) + F_V(i)$ as well as $(c,s) \to u(c) + F_V(s-c)$ have increasing differences on a Veinott's strong set order increasing set $[0,s]$. As a result, by Topkis (1978) Theorem, the solution of the right hand side of the problem (6), i.e. $c^*(s) = BV(s)$ as well as $i^*(s) = s - BV(s)$ are increasing on S. As

$$TV(s) = u(BV(s)) + \delta \int_S V(s')Q(ds'|s - BV(s)),$$

we obtain $s \to TV(s)$ is also increasing.

Next, by Theorem 1, $T^n(\mathbf{0}) \rightrightarrows V^*$ hence V^* is increasing. By the above arguments, similarly $BV^*(s)$ and $s - BV^*(s)$ are increasing functions of s. As a result, by the Lebesgue Theorem (see Theorem 17.12 in Hewitt and Stromberg, 1965), there is a Lebesgue null set N, such that for $s \in S \setminus N$, V^* and BV^* have a finite derivative. $\square$

Next, to assure that Markovian equilibrium strategy and value are differentiable on $(0,\infty)$ we need the following assumption.

Assumption 3. Assume that

- u is twice continuously differentiable. Moreover for any increasing $V \in \mathcal{V}$ we have:
- function F_V is twice continuously differentiable on $S \setminus \{0\}$;
- $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{i\to 0} F'_V(i) = \infty$,
- $|u''(s)| > 0$ or $|F''_V(s)| > 0$ for any $s \in S$.

Clearly, the Inada type conditions are assumed to obtain interior solution. The last assumption is necessary to use implicit function theorem. Clearly, both are satisfied e.g. by a power utility function. The next remark discusses the class of stochastic transitions that satisfy our conditions.

[8] In fact we can weaken our assumptions, so that Assumption 1 holds for any increasing $V \in \mathcal{V}$.

Remark 1. A class of measures Q satisfying Assumption 3 was provided by Amir (1996), i.e. $Q(\cdot|i) = (1 - g(i))\eta_1(\cdot) + g(i)\eta_2(\cdot)$ for twice continuously differentiable function $g : S \to [0, 1]$ satisfying Inada condition. In particular, a class of transitions satisfying additionally Assumption 1 can be a special case allowing η_1 to be a delta Dirac concentrated at point 0. Finally, Amir (1997) characterizes a class of measures Q satisfying Assumption 3, if associated cdf $q(s|i)$ is twice continuously differentiable with i with integrable derivatives for any $s \in S$.

Theorem 4. *Under Assumption 1 and 3 pure strategy Markovian equilibrium policy h^* and value V^* are differentiable on $(0, \infty)$.*

Proof. Fix arbitrary $s_0 > 0$. We now show that $AV^{*\prime}(s_0)$ exists. By Lemma 8 and Assumption 3, $i^*(s_0) < s_0$. Choose $\epsilon > 0$ such that $\Gamma := [i^*(s_0 - \epsilon), i^*(s_0 + \epsilon)] \subset [0, s_0 - \epsilon] \subset S$. As i^* is increasing, we have $i^*(s) \in \Gamma := [i^*(s_0 - \epsilon), i^*(s_0 + \epsilon)]$, whenever $s \in [s_0 - \epsilon, s_0 + \epsilon]$. Moreover, we can construct ϵ such that $i^*(s_0 - \epsilon) > 0$; hence, we have

$$u'(s - i^*(s)) - F'_{V^*}(i^*(s)) = 0.$$

Put $L'_1(s, i) = u'(s - i) - F'_{V^*}(i)$, and observe that

$$\inf_{(s,i)\in[s_0-\epsilon,s_0+\epsilon]\times\Gamma} \frac{\partial}{\partial i} L'_1 = \inf_{(s,i)\in[s_0-\epsilon,s_0+\epsilon]\times\Gamma} (-u''(s - i) - F''_{V^*}(i)) > 0.$$

Then, by the implicit function theorem, we obtain differentiability of BV^* and consequently V^* and TV^* at s_0. $\square$

Similar to Harris and Laibson (2001) or Judd (2004), we can now write the generalized Euler equations characterizing pure strategy Markovian equilibrium investment i. For this reason, suppose i^* is a differentiable investment equilibrium, i.e. $i^*(s) = s - h^*(s)$. To simplify notation we drop $*$ from V^* and i^*. Additionally, by $q(\cdot|i)$ denote a cdf associated with measure Q (such that Assumption 3 is satisfied). We have the following version of the Euler equation in a time-consistent Markov equilibrium:

$$u'(s - i(s)) = \beta\delta \frac{d}{di} \int_S V(s')dq(s'|i(s)), \tag{7}$$

where we have:

$$V'(s) = u'(s - i(s))(1 - i'(s)) + \delta i'(s) \frac{d}{di} \int_S V(s')dq(s'|i(s)). \tag{8}$$

Using the Fundamental Theorem of the Integral Calculus for Riemann–Stieltjes integrals (see Hewitt and Stromberg, 1965 Theorem 18.19 or Amir, 1997, Theorem 3.2), we have the envelope for $V(s)$:

$$\frac{d}{di} \int_S V(s')dq(s'|i) = -\int_S V'(s')q'(s'|i)ds',$$

where $q'(s'|i) = \frac{d}{di}q(s'|i)$. We can integrate equation (8) to arrive at:

$$\int_S V'(s)q'(s|x)ds = \int_S u'(s-i(s))(1-i'(s))q'(s|x)ds$$
$$+ \delta \int_S i'(s) \left[\frac{d}{di} \int_S V(s')dq(s'|i(s)) \right] q'(s|x)ds.$$

Let $I(x) := \frac{d}{di} \int_S V(s')dq(s'|x)$. Then, we have additionally

$$-I(x) = \int_S u'(s-i(s))(1-i'(s))q'(s|x)ds + \delta \int_S I(i(s))i'(s)q'(s|x)ds.$$

From equation (7):

$$-I(x) = \int_S u'(s-i(s))(1-i'(s))q'(s|x)ds + \frac{1}{\beta} \int_S u'(s-i(s))i'(s)q'(s|x)ds.$$

Therefore, the generalized Euler equation can be obtained by rewriting equation (7) as:

$$\begin{aligned} u'(x-i(x)) &= -\beta\delta \int_S u'(s-i(s))(1-i'(s))q'(s|i(x))ds \\ &\quad - \delta \int_S u'(s-i(s))i'(s)q'(s|i(x))ds \\ &= -\beta\delta \int_S u'(s-i(s))[1+(\frac{1}{\beta}-1)i'(s)]q'(s|i(x))ds. \end{aligned}$$

The above equation is a stochastic counterpart of the generalized Euler equation in Harris and Laibson (2001) or Judd (2004). Recall, our application of Lebesgue differentiation theorem for Riemann–Stieltjes integrals is satisfied for absolutely continuous functions, a class including functions of bounded variation studied in the original construction of the generalized Euler equation by Harris and Laibson (2001). We further relate this result with the existing literature in the last section of the paper.

3. Relating the results to the literature

Equilibrium non-existence and/or multiplicity of equilibria have constituted a significant challenge for applied economists who sought to study models where dynamic consistency failures play a key role (see e.g. Maliar and Maliar, 2016). These issues have been equally as challenging for researchers that seek to identify tractable numerical approaches to computing Markovian equilibria in these (and related) dynamic games (e.g., see the discussion in Krusell and Smith, 2003 or Judd, 2004).

Krusell et al. (2002) propose a generalized Euler equation method for a version of a hyperbolic discounting consumer, and additionally obtain explicit solution for logarithmic utility and Cobb–Douglas production examples. Per the latter result, of course, this is simply an example, which is well-known to not be robust to small variations of the primitive data of the economy. In Harris and Laibson (2001) and Judd (2004), the author's propose a generalized Euler equation approach to analyze smooth time-consistent policies and proposes a perturbation method for calculating them. The problem with this approach is providing sufficient conditions under which at any point in the state space, the generalized Euler equations represent a *sufficient* first order theory for an agent's value function in the equilibrium of the game. Concentrating on non-smooth policies, Krusell and Smith (2003) define a step function equilibrium, and show its existence and resulting indeterminacy of steady state capital levels. Further, in a deterministic setting, general existence result of optimal policies under quasi-geometric discounting can be provided using techniques proposed by Goldman (1980). The problem raised by these results concern the multiplicity and/or indeterminacy of dynamic equilibrium, which makes the structure of the set of equilibrium of such models very difficult to characterize (hence, the models are difficult to use in applied work).

To circumvent some of this issues, many authors have added noise to agent decision problems, and/or studied the existence of time-consistent solutions in relevant dynamic game formulations. Specifically, in a *(recursive) decision approach*, by adding noise (making payoff discontinuities negligible) Caplin and Leahy (2006) prove existence of recursively optimal plan for a finite horizon decision problem and general utility functions. Similarly, Bernheim and Ray (1986) show that by adding enough noise to the dynamic game (to smooth discontinuities away) existence of strategy Markovian equilibrium is guaranteed. Such *stochastic game approach* was later developed by Harris and Laibson (2001) who characterize the set of smooth Markovian equilibrium by (generalized) first order conditions.

In the related paper Balbus et al. (2015) propose a similar stochastic game method for studying pure strategy Markovian equilibrium policies of the more general quasi-hyperbolic discounting game. Based on their generalization of the Tarski–Kantorovich fixed point theorem, they are able to show existence of the equilibria for the case of *bounded* returns in a wide range of problems, and provide successive approximation procedures that compute extremal equilibrium values. Unfortunately, the question of approximating equilibrium that *support* such equilibrium values remains a substantial problem in their work. In this paper, we provide new sufficient conditions under which *unique* Markovian equilibria policies can also be computed associated with *unique* equilibrium values via a simple generalized Bellman method. Further, per the question of existence, we relax conditions on the boundedness of period return functions.

Additionally, recently Balbus and Woźny (2016) provided an APS type method for analyzing non-stationary Markovian policies of the quasi-hyperbolic discounting game numerically using set approximation techniques. One issue with this method is its inability to characterize the set of non-stationary Markovian policies that support the equilibrium value correspondence in the game. In principle, our results in this paper are also related to strategic dynamic programming/APS approaches (as the latter approaches amount to set-valued dynamic programming methods). If the methods of Balbus and Woźny (2016) are applied to the environment studied in this paper, they would converge to the unique equilibrium value associated with a pure strategy Markovian equilibrium.

Finally, in an interesting recent paper, Chatterjee and Eyigungor (2016) prove a existence result in randomized Markovian strategies, and discuss when such equilibria exist in a class of continuous functions. As compared to our paper, note that apart from differences in assumptions (endogenous vs. exogenous concavification of the expected value function), our results differ in many important dimensions. First, our existence result concern *pure strategies*, rather then randomized policies. Second, our uniqueness result is satisfied relative to a wide class of bounded, measurable value functions, not just *continuous* values. This fact, when added with a version of our existence result proven in Balbus et al. (2015) (Theorem 5) can be used to show existence of continuous (pure) Markovian equilibrium. Finally, notice our assumption on stochastic transition probability for the game requires an atom at zero asset level. This condition has a flavor of the nonexistence of a lower bound of wealth, the assumption that was shown by Chatterjee and Eyigungor (2016) to be a critical source of problems with continuous (pure) Markovian equilibrium existence.

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