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A Strategic Dynamic Programming Method for Studying Short-Memory Equilibria of Stochastic Games with Uncountable Number of States

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Abstract We study a class of infinite horizon stochastic games with uncountable number of states. We first characterize the set of all (nonstationary) short-term (Markovian) equilibrium values by developing a new (Abreu et al. in Econometrica 58(5):1041–1063, 1990)-type procedure operating in function spaces. This (among others) proves Markov perfect Nash equilibrium (MPNE) existence. Moreover, we present techniques of MPNE value set approximation by a sequence of sets of discretized functions iterated on our approximated APS-type operator. This method is new and has some advantages as compared to Judd et al. (Econometrica 71(4):1239–1254, 2003), Feng et al. (Int Econ Rev 55(1):83–110, 2014), or Sleet and Yeltekin (Dyn Games Appl doi:10.1007/s13235-015-0139-1, 2015). We show applications of our approach to hyperbolic discounting games and dynamic games with strategic complementarities.

Keywords Stochastic games · Hyperbolic discounting · Supermodular games · Short-memory (Markov) equilibria · Constructive methods · Computation · Approximation

JEL Classification C72

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1 Introduction and Related Literature

The existence of equilibrium in the class of discounted, infinite horizon stochastic games is an important question since the work of Shapley [50] and includes many seminal contributions (see e.g., [40] or [47]). Recently, however, economists focused on equilibrium existence in the class of short-memory strategies, especially over uncountable number of states. The importance of short-memory equilibria lays in (i) its simplicity, (ii) potential computational tractability, (iii) possibility of developing methods for studying comparative statics or dynamics, and as a result, (iv) applicability to many economic problems at hand, among others. This literature has many recent contributions, including work of Duggan [28], who proves equilibrium existence in a class of noisy stochastic games, of Barelli and Duggan [16] and Jaśkiewicz and Nowak [33], who focus on semi- or almost-Markov equilibria, or of Balbus et al. [9] with few general stationary equilibrium existence results, or of Levy and McLennan [39], who show nonexistence of stationary equilibrium in a class of continuous stochastic games, among others.

Further, in recent times, stochastic games have become a fundamental tool for studying dynamic economic models, where agents possess some form of limited commitment including works in (i) dynamic political economy, (ii) equilibrium models of stochastic growth without commitment, (iii) models of savings and asset prices with hyperbolic discounting, (iv) international lending and sovereign debt, (v) optimal Ramsey taxation, (vi) dynamic negotiations with status quo, or (vii) dynamic oligopoly models, for example. The applications of repeated, dynamic or stochastic games toolkit to analyze these phenomena results, among others, from the richness of behavior supported by a subgame perfect or sequential equilibrium (see celebrated folk theorem and its analytical tractability using recursive characterization of public equilibria of pathbreaking APS [2] contribution).

Additionally, in the literature pertaining to economic applications of stochastic games, the central concerns have been broader than the mere question of weakening conditions for the existence of subgame perfect or Markovian equilibrium. Rather, researchers have become more concerned with characterizing the properties of computational implementations, so they can study the quantitative (as well as qualitative) properties of particular subclasses of perfect equilibrium. Unfortunately, for uncountable number of states, there are only few papers and results that offer a set of rigorous tools to compute, approximate or characterize the equilibrium strategies. See [12–14] for some recent contributions.

The aim of this paper is to address the question of short-memory equilibrium existence, characterization and computation using constructive monotone method, where our notion of monotonicity is defined using set inclusion order over spaces of value functions. Specifically, we study existence and approximation, relative to the set of MPNE for two important classes of stochastic games, namely: (i) hyperbolic discounting games[1] and (ii) supermodular games with strategic (within period) complementarities and positive externalities,[2] although applications in other classes of games are possible (see [11], for an early example applied to OLG economies).

From this perspective, the contribution of our paper is as follows. We first prove existence of a *Markovian* NE via strategic dynamic programming methods similar to that proposed in the seminal work of Mertens and Parthasarathy [40][3] and APS [2] (henceforth, MP/APS). We refer to this as a "indirect" method, as these methods focus exclusively on equilibrium values

[1] As analyzed in [8,10,31,32,46] or [15], for example.

[2] As analyzed in [5,26,45] or [14], for example.

[3] See also [41] or [42], pages 397–398.

(rather than, characterizing the set of strategies that implement those equilibrium values). To mention, our method differs from those of the traditional MP/APS literature in at least two directions. Perhaps most importantly, we study the existence of short-memory or Markovian equilibria, as opposed to broad classes of sequential or subgame perfect equilibria.[4] Additionally, our strategic dynamic programming method works directly in function spaces (as opposed to spaces of correspondences), by which we can avoid some of the technical problems associated with measurability and numerical implementations using set-approximation techniques.[5]

Next, we propose a procedure for MPNE value set approximation. This differs from the approach taken by Judd et al. [34] and Feng et al. [30] (henceforth, FMPS) or Sleet and Yeltekin [51] as: (i) our theoretical numerical method operates directly in function spaces, (ii) we allow to analyze equilibria that are time/state dependent only (but are not continuation dependent), and moreover, (iii) equilibria we study are defined on a minimal state space, which greatly simplifies the approximation of the set of policies that implement particular values in the equilibrium value set.

The rest of the paper is organized as follows. We start in Sect. 2 by presenting and discussing our method. Then, in Sect. 3, we present application to a class of quasi-hyperbolic discounting model. Next, our results on Markov equilibrium existence and its value set approximation for a class of stochastic supermodular games can be found in Sect. 4. Section 5 concludes.

2 The Method

The approach we take in this paper to prove existence of MPNE and approximate its value set, is the strategic dynamic programming approach based on the seminal work of Mertens and Parthasarathy [40]. See also [1,2] per similar method adapted for repeated games. In the original strategic dynamic programming approach, dynamic incentive constraints are handled using correspondence-based arguments. For each state $s \in S$, one shall envision agents playing a one-shot stage game with the continuation structure parameterized by a measurable correspondence of continuation values, say $w \in \mathcal{V}$, where $\mathcal{V}$ is the space of nonempty, bounded, upper semicontinuous correspondences (for example). Imposing incentive constraints on deviations of the stage game under some continuation promised utility w, an operator B, that is monotone under the set inclusion order, can be defined that transforms $\mathcal{V}$. By iterating on B from the greatest element of $\mathcal{V}$, the operator is shown to map down, and then, by appealing to standard "self-generation" arguments, it can be shown a descending subchain of subsets can be constructed, whose pointwise limit in the Hausdorff topology is the greatest fixed point of B. This fixed point turns out to be the set of sustainable values in the game, with a subgame perfect/sequential equilibrium being any measurable mapping supporting measurable selections from this limiting correspondence of values.

[4] It bears mentioning, we focus on short-memory Markovian equilibrium because this class of equilibrium has been the focus of a great deal of applied work. We should also mention very interesting papers by Cole and Kocherlakota [24] and Doraszelski and Escobar [27] that also pursue a similar idea of trying to develop MP/APS-type procedure in function spaces for Markovian equilibrium (i.e., methods where continuation structures are parameterized by functions) but for finite/countable number of states. See also [18] for a related argument used to prove existence of equilibrium in a bequest game.

[5] In our accompanied papers (see [11,14,15]), we propose an alternative *direct* method for *stationary* Markov equilibrium existence and computation. We view results of both, direct and indirect, methods as complementary and discuss them in the paper.

In this paper, we propose a new procedure for constructing *all* measurable (possibly nonstationary) Markov Nash equilibria for a class of infinite horizon stochastic games. Our approach[6] is novel, as we operate directly in *function spaces*,[7] i.e., a set of bounded measurable functions on S valued in $\mathbb{R}$.

To see this, consider an auxiliary strategic form game G_w^s parameterized by a continuation w and state $s \in S$. By $T(w)(s)$ denote a set of all Nash equilibrium payoffs of G_w^s, measurable in s. Then, we define an operator $B : \mathcal{V} \to \mathcal{V}$, where $\mathcal{V} := 2^V$ for some compact set V, a subset of a class of bounded, measurable functions. We let:

$$B(W) = \cup_{w \in W} \{v \in V : \forall s \in S \; v(s) = T(w)(s)\}.$$

Operator B is nondecreasing on $\mathcal{V}$, when endowed with the set inclusion order. By $V^* \subset V$, we denote a set of all values in V for some MPNE. In the next sections, we derive conditions under which the following result holds: $\cap_t B^t(V) = V^* \neq \emptyset$ (t-th composition). We also provide a method of V^* approximation in the Hausdorff distance. Before proceeding, we discuss some important properties of our method and discuss its relation to the literature.

First, let us address the differences between the traditional correspondence-based MP/APS procedure and our's. Let V_{APS}^* be the correspondence of sequential equilibria satisfying $B_{APS}(V_{APS}^*) = V_{APS}^*$. From the definition of MP/APS operator, we know the following:

$$(\forall s \in S) \left(\forall \text{ vector } v \in V_{APS}^*(s)\right) \left(\exists \text{ measurable function } w \in V_{APS}^* \text{ s.t. } v = T(w)(s)\right).$$

Specifically, observe that continuation function w can depend on v and s, hence we shall denote it by $w_{v,s}$. Now, consider our operator B and its fixed point $V^* \subset V$. We have the following property:

$$(\forall \text{ function } v \in V^*) \left(\exists \text{ measurable function } w \in V^* \text{ s.t. } (\forall s \in S)\, v(s) = T(w)(s)\right).$$

In our method, continuation w depends on v only; hence, we can denote it by w_v.

Observe that in both methods, the profile of equilibrium decision rules: $NE(w, s)$ is generalized Markov, as it is enough to know state s and continuation function w to make an optimal choice. In our technique, however, the dependence of v on the current state is direct: $s \to NE(s, w_v)$. So we can verify whether the generalized Markov policy is, e.g., continuous, monotone in s easily. In the MS/APS approach, one has the following: $s \to NE(s, w_{v,s})$, so even if NE is continuous in both variables, there is no way (generally) to control continuity of $s \to w_{v,s}$. The best example of such discontinuous continuation selection is, perhaps, the time-consistency model (see [21]). These problems are also the main motivation for developing a computational technique that uses specific properties of (the profile) of the equilibrium decision rules with respect to s (important especially, when the state space is uncountable).

On a related matter, the set of Markov perfect Nash equilibrium values is a subset of the MP/APS (subgame perfect/sequential) equilibrium value set.

This framework used to prove existence of the MPNE can be also used to define a piecewise constant approximation of the equilibrium set V^*. Our theoretical numerical method is directly linked to the proof of equilibrium existence, hence heavily relies on our theoretical result, i.e., it uses the fact that (i) our method operates directly in function spaces, (ii) allows

6 It bears mentioning that for dynamic games with more restrictive shocks spaces (e.g., discrete or countable), MP/APS procedure has been used extensively in economics in recent years: see e.g., [18] for altruistic economies, [7,37,48,49] for policy games or FMPS for recursive competitive equilibrium of a dynamic economy.

7 For example, see Phelan and Stacchetti, who discuss such possibility in function spaces.

to analyze equilibria that are time/state dependent only and (iii) studies equilibria defined on a minimal state space. All of these greatly simplify the approximation of the set of policies that implement particular values in the equilibrium value set. The details are presented for the two examples we study in the next sections.

3 A Class of Quasi-hyperbolic Discounting Games

Our environment is a stochastic version of a $\beta - \delta$ quasi-hyperbolic discounting model that has been studied extensively in the literature[8] (see [31] or [15]). We envision an agent to be a sequence of "selves" indexed in discrete time $t \in T = \{0, 1, \ldots\}$. A "current self" or "self t" enters the period in given state $s_t \in S$, where for some $\bar{S} \in \mathbb{R}_+$, $S := [0, \bar{S}]$, and chooses an action denoted by $c_t \in [0, s_t]$. This choice determines a stochastic transition probability on the next period state s_{t+1} given by $Q(ds_{t+1}|s_t - c_t)$. The within-period utility is given by (bounded) utility function u. Discount factor from today (t) to tomorrow ($t + 1$) is $\beta\delta$, but equals δ between any two future dates $t + s$ and $t + s + 1$ for $s > 0$. Thus, preferences (discount factor) depend on date s.

We now define preferences and a MPNE for the quasi-hyperbolic consumer:

Definition 1 $h := (h_t)_{t\in\mathbb{N}}$ is a MPNE, if there is a sequence $(v_t)_{t\in\mathbb{N}}$, where each v_t is integrable, such that for each $t \in \mathbb{N}$ and $s \in S$

$$h_t(s) \in \arg\max_{c\in[0,s]} \left\{(1-\beta)u(c) + \beta\delta \int_S v_{t+1}(s')Q(ds'|s-c)\right\}.$$

and

$$v_t(s) = (1-\beta)u\left(h_t(s)\right) + \delta \int_S v_{t+1}(s')Q(ds'|s-h_t(s)).$$

Here, for uniformly bounded v_t, we have

$$v_t(s) = J\left((h_\tau)_{\tau=t}^{\infty}\right)(s) := (1-\beta)E_s^{h,t}\left(\sum\nolimits_{\tau=1}^{\infty} \delta^{\tau-1}u(h_{t+\tau-1})\right),$$

where $E_s^{h,t}$ is an expectation operator with respect to the unique probability measure on the set of all histories induced by integrable h and state s at stage t. Intuitively, a current self best responds to the value v_{t+1} discounted by $\beta\delta$ that summarizes payoffs from future "selfs" strategies $(h_\tau)_{\tau=t+1}^{\infty}$, and such best response h_t is used to update v_{t+1} discounted by δ to v_t.

3.1 Existence and Characterization

For given $\bar{S} \in \mathbb{R}_+$, $S = [0, \bar{S}]$ define a function space:

$$V := \left\{v : S \to \mathbb{R}_+ : v \text{ is nondecreasing and u.s.c. bounded by } u(0) \text{ and } u(\bar{S})\right\}.$$

endowed with the weak topology. See e.g., [29] for a formal definition of this topology. Here we only note that V endowed with the weak topology is a compact set. Also, weak topology restricted to V is metrizable. We say that a set is *weakly compact*, if it is compact in the weak topology. In this section, $\Rightarrow$ denotes weak convergence, $\to^u$ means uniform convergence, while $\to$ means pointwise convergence.

8 See also [22], who use APS technique to analyze equilibria of a n-player quasi-hyperbolic discounting game with imperfect monitoring.

Let CM be a set of nondecreasing, Lipschitz continuous (with modulus 1) functions $h : S \to S$, such that $\forall s \in S\, h(s) \in [0, s]$. Clearly, CM is a compact set, when endowed with the topology of uniform convergence. Let:

$$V^* = \{v \in V : \exists \text{ MPNE } (h_t)_{t\in\mathbb{N}}, \text{ where each } h_t \in CM, \text{ s.t } v(s) = J((h_t)_{t\in\mathbb{N}})(s)\, \forall s \in S\}.$$

Assumption 1 Assume that:

- $u : S \to \mathbb{R}$ is continuous, increasing and strictly concave,
- for each $v \in V$, function $i \to \int_S v(s')Q(ds'|i)$ is continuous, increasing and concave,
- for each $i \in S$, $Q(\cdot|i)$ is a nonatomic measure.

An example[9] of transition Q that satisfies Assumption 1 is

$$Q(\cdot|i) = g(i)\lambda_2(\cdot) + (1 - g(i))\lambda_1(\cdot),$$

with probability measure λ_2 first-order stochastically dominating probability measure λ_1 for some continuous, increasing and concave $g : S \to [0, 1]$. Put:

$$\Pi^\kappa(c, s, v) := (1 - \beta)u(c) + \kappa \int_S v(s')Q(ds'|s - c)$$

for $\kappa \in [0, 1]$, and define an operator B on 2^V by:

$$B(W) := \bigcup_{w\in W} \Big\{ v \in V : (\forall s \in S)\, v(s) = \Pi^\delta(h(s), s, w), \text{ for some } h : S \to S,$$
$$\text{s.t. } h(s) \in \arg\max_{c\in[0,s]} \Pi^{\beta\delta}(c, s, w) \text{ for all } s \in S \Big\}.$$

We start by few lemmas.

Lemma 1 *Suppose $h \in CM$ and $\kappa \in [0, 1]$. Define A_h^κ as follows*

$$A_h^\kappa(v)(s) = \Pi^\kappa(h(s), s, v).$$

Then, $A_h^\kappa : V \to V$.

Proof of lemma 1 Let $v \in V$ and $h \in CM$. Then, both $h(s)$ and $s - h(s)$ are nondecreasing in s. By Assumption 1, $A_h^\kappa(v)(\cdot)$ is nondecreasing. Since $v \in V$ and h is continuous, $A_h(v)^\kappa(\cdot)$ is continuous. Therefore, A_h^κ is a self-map on V. □

Lemma 2 *Let $(v_n)_{n\in\mathbb{N}}$, where each $v_n \in V$ and $(h_n)_{n\in\mathbb{N}}$, where each $h_n \in CM$.*

(i) *Suppose $v_n \Rightarrow v$ and $h_n \to^u h$ as $n \to \infty$. Then,*

$$\Pi^\kappa(h_n(s), s, v_n) \to \Pi^\kappa(h(s), s, v), \tag{1}$$

pointwise in $s \in S$.

(ii) *If $v_n \to^u v$ and $h_n \to^u h$, then convergence in (1) is uniform.*

[9] See also [4] or [44] for related assumptions on the transition probability.

Proof of lemma 2 Proof of (i). First assume $v_n \Rightarrow v$. By Theorem 5.5 in [20], we have:

$$\int_S v_n(s')Q(ds'|s-h_n(s)) \to \int_S v(s')Q(ds'|s-h(s)). \tag{2}$$

The remaining part of the proof of (i) follows immediately from Assumption 1.

Proof of (ii). Suppose $v_n \to^u v$ and $h_n \to^u h$, then by Assumption 1, and Theorem 5.5 in [20], we have convergence in (2) for all $s \in S$. We now show that this is the uniform convergence. Put $q_v(\cdot) := \int_S v(s')Q(ds'|\cdot)$. Let $\epsilon > 0$ be given and choose $\epsilon_0 > 0$ a real value such that $|q_v(i_1) - q_v(i_2)| < \frac{\epsilon}{2}$ whenever $|i_1 - i_2| < \epsilon_0$. Choose $n_0 \in \mathbb{N}$ s.t. $||v_n - v||_\infty < \frac{\epsilon}{2}$ and $||h_n - h||_\infty < \epsilon_0$ for all $n \geq n_0$, where $||\cdot||_\infty$ is the sup norm. Then, for such n, we have

$$\begin{aligned}
&\left|\int_S v_n(s')Q(ds'|s-h_n(s)) - \int_S v(s')Q(ds'|s-h(s))\right| \\
&\leq \int_S |v_n(s') - v(s')|Q(ds'|s-h_n(s)) + |q_v(s-h_n(s)) - q_v(s-h(s))| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

for each $s \in S$. Hence, a convergence in (2) and (1) is uniform. □

Lemma 3 *Let $v \in V$ and $\kappa \in [0,1]$. Then,*

$$c_v^\kappa(s) := arg \max_{c \in [0,s]} \Pi^\kappa(c,s,v) \tag{3}$$

is a well-defined function. Moreover, $c_v^\kappa \in CM$.

Proof of lemma 3 Let $v \in V$ and $\kappa \in [0,1]$. Obviously $A(s) = [0,s]$ is Veinott strong set order increasing. Observe that by Assumption 1 function $(c,s) \to (1-\beta)u(c) + \kappa \int_S v(s')Q(ds'|s-c)$ has increasing differences. Moreover: $(i,s) \to (1-\beta)u(s-i) + \kappa \int_S v(ds')Q(ds'|i)$ has also increasing differences, and the correspondence in (3) is well defined. Hence, by Theorem 6.2 in [54], c_v^κ is well defined and $c_v^\kappa \in CM$. □

Lemma 4 *Let $\kappa \in [0,1]$ be arbitrary. Let $(v_n)_{n\in\mathbb{N}}$, where each $v_n \in V$. Then, if $v_n \Rightarrow v$, then*

$$c_{v_n}^\kappa \to^u c_v^\kappa. \tag{4}$$

Proof of lemma 4 Suppose $v_n \Rightarrow v$ on V. By Lemma 3, $(c_{v_n}^\kappa)_{n\in\mathbb{N}}$, where each $c_{v_n}^\kappa \in CM$. We only need to show convergence in (4) pointwise in s. Since CM is compact, we may suppose $c_{v_n}^\kappa \to^u c^*$. Take an arbitrary $c \in [0,s]$. Since

$$\Pi^\kappa(c_{v_n}^\kappa(s), s, v_n) \geq \Pi^\kappa(c,s,v_n),$$

hence, by Lemma 2, we have

$$\Pi^\kappa(c^*(s), s, v) \geq \Pi^\kappa(c,s,v).$$

Therefore, by Lemma 3, $c^*(s) = c_v^\kappa(s) = \lim_{n\to\infty} c_{v_n}^\kappa(s)$. □

Lemma 5 *We can express operator B as follows:*

$$B(W) = \bigcup_{w\in W} \left\{v \in V,\ (\forall s \in S)\, v(s) = \Pi^\delta(c_w^{\beta\delta}(s), s, w)\right\}. \tag{5}$$

Proof of lemma 5 Equation (5) follows immediately from definition of B and Lemma 3. □

Lemma 6 *If W is compact (in the topology of uniform convergence), then $B(W)$ is compact (in the topology of uniform convergence).*

Proof of lemma 6 Let $(v_n)_{n\in\mathbb{N}}$, where each $v_n \in B(W)$. From Lemma 5, B satisfies (5). Then, for each $n \in \mathbb{N}$, there is $w_n \in W$ s.t.

$$v_n(s) = \Pi^\delta \left(c^{\beta\delta}_{w_n}(s), s, w_n\right) \tag{6}$$

for each $s \in S$. Since W is compact, without loss of generality suppose $w_n \to^u w$ (as $n \to \infty$) for some $w \in W$. From Lemma 4 $c^{\beta\delta}_{w_n} \to^u c^{\beta\delta}_w$. Hence, by Lemma 2:

$$v_n(s) = \Pi^\delta(c^{\beta\delta}_{w_n}(s), s, w_n) \to^u \Pi^\delta(c^{\beta\delta}_w(s), s, w) := v(s).$$

Therefore, $v_n \to^u v$. Since $w \in W$, hence $v \in B(W)$. □

Lemma 7 *If $W \subset B(W)$, then $W \subset V^*$.*

Proof of lemma 7 By Lemma 5, B satisfies (5). Let $v \in W$. Then, $v \in B(W)$, and there is $w_1 \in W$ such that $v(s) = \Pi^\delta\left(c^{\beta\delta}_{w_1}, s, w_1\right)$ for each $s \in S$. Consequently $w_1 \in W$ implies $w_1 \in B(W)$, and hence $w_1(s) = \Pi^\delta\left(c^{\beta\delta}_{w_2}(s), s, w_2\right)$ for all $s \in S$ and for some $w_2 \in W$. We continue this procedure and receive $(w_n)_{n\in\mathbb{N}}$, where each $w_n \in W$ such that $w_n(s) = \Pi^\delta\left(c^{\beta\delta}_{w_{n+1}}(s), s, w_{n+1}\right)$ for all $n \in \mathbb{N}$ and $v(s) = \Pi^\delta(c^{\beta\delta}_{w_1}, s, w_1)$. By definition of Π^δ, $c^{\beta\delta}_{w_1}$ and V^* we have $v \in V^*$. Hence, $W \subset V^*$. □

Lemma 8 *$B(V)$ is compact (in the topology of uniform convergence).*

Proof of lemma 8 We show that $B(V)$ is equicontinuous. Let $(s^n_1)_{n\in\mathbb{N}}$, and $(s^n_2)_{n\in\mathbb{N}}$, where each $s^n_1 \in S$ and $s^n_2 \in S$, be chosen s.t. $|s^n_1 - s^n_2| \to 0$ as $n \to \infty$. By (5), we need to show that

$$\sup_{w\in V} \left|\Pi^\delta\left(c^{\beta\delta}_w(s^n_1), s^n_1, w\right) - \Pi^\delta\left(c^{\beta\delta}_w(s^n_2), s^n_2, w\right)\right| \to 0 \quad (n \to \infty). \tag{7}$$

Choose arbitrary $\epsilon > 0$. Let $(w_n)_{n\in\mathbb{N}}$, where each $w_n \in V$ be such that $\Delta(w_n)(s^n_1, s^n_2) > \Delta(s^n_1, s^n_2) - \epsilon$, where

$$\Delta(w)(s_1, s_2) := |\Pi^\delta\left(c^{\beta\delta}_w(s_1), s_1, w\right) - \Pi^\delta\left(c^{\beta\delta}_w(s_2), s_2, w\right)|$$

and $\Delta(s_1, s_2) = \sup_{w\in V} \Delta(w)(s_1, s_2)$ for each $s_1, s_2 \in S$. Since V includes increasing, u.s.c. functions and commonly bounded from above, hence we may assume that $w_n \Rightarrow w$ for some $w \in V$. Observe that by Theorem 5.5 in [20]:

$$\int_S w_n(s')Q(ds'|s^n_1 - c^{\beta\delta}_{w_n}(s^n_1)) - \int_S w_n(s')Q(ds'|s^n_2 - c^{\beta\delta}_{w_n}(s^n_2)) \to 0 \quad (\text{as } n \to \infty). \tag{8}$$

We also have $|u(c^{\beta\delta}_{w_n}(s^n_1)) - u(c^{\beta\delta}_{w_n}(s^n_2))| \to 0$, whenever $n \to \infty$. Hence, $\Delta(w_n)(s^n_1, s^n_2) \to 0$. As a result, $\limsup_{n\to\infty} \Delta(s^n_1, s^n_2) < \epsilon$. Since $\epsilon > 0$ is arbitrary, hence (7) holds. As a result, $B(V)$ is equicontinuous in the topology of uniform convergence and includes functions bounded by some common value. Hence, by Arzela–Ascoli theorem, it is a compact set in the topology of uniform convergence. □

Theorem 1 *Assume 1. Then:*

(i) $\bigcap_{t=1}^{\infty} B^t(V) \neq \emptyset$,

(ii) $\bigcap_{t=1}^{\infty} B^t(V)$ *is the greatest fixed point of* B,

(iii) $\bigcap_{t=1}^{\infty} B^t(V) = V^*$.

Proof of theorem 1 We show (i). By Lemma 8, $B(V)$ is a compact set, and hence, by Lemma 6, for all $t \in \mathbb{N}$ $B^t(V)$ is compact and nonempty set. As a result, $\bigcap_{t\in\mathbb{N}} B^t(V) \neq \emptyset$. To show (ii) observe that as $\mathcal{V}$ is a complete lattice, B is nondecreasing, and by Tarski Theorem, B has the greatest fixed point, say W^*. Moreover, as B is nondecreasing, $\{W_t\}_{t=0}^{\infty}$, where $W_t = B^t(V)$, is a descending sequence (under set inclusion). We need to show that $\bigcap_{t=1}^{\infty} W_t = W^*$. Clearly, $\bigcap_{t=1}^{\infty} W_t \subset W_t$ for all $t \in \mathbb{N}$; hence:

$$B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_{t+1} = \bigcap_{t=1}^{\infty} W_t.$$

To show equality, it suffices to show $\bigcap_{t=1}^{\infty} W_t \subset B(\bigcap_{t=1}^{\infty} W_t)$. Let $w \in \bigcap_{t=1}^{\infty} W_t$. Then, $w \in W_t$ for all t. By the definition of W_t and B, we obtain existence of the sequence $v^t \in W_t$ and best response $h^t \in CM$ such that $w(s) = \Pi^\delta(h^t, s, v^t)$ for all s and t. Since $B(V)$ is compact, without loss of generality, assume v^t converges uniformly to v^*. Moreover, $v^* \in \bigcap_{t=1}^{\infty} W_t$, since W_t is a descending set of compact sets. By Lemma 4, $h^* = c_{v^*}^{\beta\delta}$. Without loss of generality, let $h^t \to h^*$. Hence, we obtain $w^* \in B(\bigcap_{t=1}^{\infty} W_t)$. Hence, $\bigcap_{t=1}^{\infty} W_t$ is a fixed point of B, and, by definition $\bigcap_{t=1}^{\infty} W_t \subset W^*$. To finish the proof, we simply need to show $W^* \subset \bigcap_{t=1}^{\infty} W_t$. Since $W^* \subset V$, $W^* = B(W^*) \subset B(V) = W_1$. By induction, we have $W^* \subset W_t$ for all t; hence, $W^* \subset \bigcap_{t=1}^{\infty} W_t$. Therefore, $W^* = \bigcap_{t=1}^{\infty} W_t$.

Now, we show (iii). We show that the right-hand side in (iii) is self-generating. Let $v \in \bigcap_{t\in\mathbb{N}} B^t(V)$. Then, for each $t \in \mathbb{N}$, there is $w_t \in B^t(V)$ such that $v(s) = \Pi^\delta(c_{w_t}^{\beta\delta}, s, w_t)$. Without loss of generality, suppose $w_t \to^u w$ for some $w \in V$. By Lemma 4, $c_{w_t}^{\beta\delta} \to^u c_w^{\beta\delta}$. By Lemma 2, we have

$$v(s) = \Pi^\delta(c_{w_t}^{\beta\delta}, s, w_t) \to \Pi^\delta(c_w^{\beta\delta}, s, w)$$

for each $s \in S$. In fact, $v(s) = \Pi^\delta(c_w^{\beta\delta}, s, w)$. Since $\left(B^t(V)\right)_{t\in\mathbb{N}}$ is a nonincreasing sequence, $w_t \in B^t(V)$ for all $t \in \mathbb{N}$, $w_t \to^u w$, and hence $w \in \bigcap_{t\in\mathbb{N}} B^t(V)$. Consequently, $v \in B\left(\bigcap_{t\in\mathbb{N}} B^t(V)\right)$. Hence, the right side in (iii) is self-generating. Therefore, by Lemma 7:

$$\bigcap_{t\in\mathbb{N}} B^t(V) \subset V^*.$$

Now, we show the reverse inclusion. Let $v^* \in V^*$. Then, $v^*(s) = \Pi^\delta\left(c^{\beta\delta}_{w_1}(s), s, w_t\right)$, where $w_t(s) = \Pi^\delta\left(c^{\beta\delta}_{w_{t+1}}(s), s, w_{t+1}\right)$ for all $s \in S$ and $t \in \mathbb{N}$. Hence, $v^* \in B^t(V)$ for each $t \in \mathbb{N}$, and $v^* \in \bigcap_{t\in\mathbb{N}} B^t(V)$. Equation in (iii) is hence satisfied. □

3.2 The Numerical Method

We now define our numerical method in details. We define $\mathbb{R}^S$ as a set of all real-valued functions with domain S.

Definition 2 A set of functions $\mathcal{F} \subset \mathbb{R}^S$ is sequentially compact if for any sequence $(f_n)_{n\in\mathbb{N}}$, where each $f_n \in \mathcal{F}$, we find a subsequence f_{n_k} of f_n and a function $f \in \mathcal{F}$ such that $f(s) = \lim_{k\to\infty} f_{n_k}(s)$ for all $s \in S$.

For each $s \in S$, we define $\pi_s : \mathbb{R}^S \to \mathbb{R}$ as follows $\pi_s(f) := f(s)$. By $d_H(A, B)$, we define a Hausdorff distance between bounded subsets $A, B \subset \mathbb{R}$. Moreover, we have:

Lemma 9 *Let Assumption 1 be satisfied. Then, each W_t $(t \geq 1)$ is a sequentially compact subset of $\mathbb{R}^S$. For each $t \in \mathbb{N}$ and $s \in S$ put $W_t(s) := \pi_s(W_t)$. Then, $\forall s \in S$:*

$$V^*(s) := \bigcap_{t=1}^{\infty} W_t(s). \tag{9}$$

Proof of lemma 9 By Lemma 16 in the Appendix, V is a sequentially compact set, and then by Lemmas 8 and 6, W_t is a compact set; hence, it is sequentially compact. As a result, for each $t \in \mathbb{N}$ and $s \in S$ the sets $W_t(s)$ are compact in a natural Euclidean topology on S, hence $\bigcap_{t\in\mathbb{N}} W_t(s) \neq \emptyset$. We now show that (9) is satisfied. By Theorem 1, we have $V^* = \bigcap_{t=1}^{\infty} W_t$. Hence, $\forall s \in S$:

$$V^*(s) \subset \bigcap_{t=1}^{\infty} W_t(s).$$

We need to show the converse inclusion. Let $s \in S$ be given and suppose $x \in V^*(s)$. Then, for each $t \in \mathbb{N}$, we find a function $f_t \in W_t$ such that $x = f_t(s)$. Since each W_t is sequentially compact, without loss of generality, we may assume $f_t \to f$. We claim that $f \in V^*$. Take arbitrary $k \in \mathbb{N}$. Then, $f_t \in W_k$ for all $t \geq k$. Since W_k is sequentially compact, $f \in W_k$. Since k is arbitrary, $f \in V^*$. Obviously $x = f(s)$. Therefore, $x = \pi_s(f) \in \pi_s(V^*)$. □

We consider an approximation of $V^*(s)$ related to [17]. Since S is an interval in $\mathbb{R}$, we can define a piecewise constant multifunction in the following way: If $S = [\xi, \eta]$, then we divide $[\xi, \eta]$ into 2^j subintervals with equal length. Let $\mathcal{C}_j$ be a block partition set. For each block partition $C \in \mathcal{C}_j$, we define θ_C as follows:

$$\theta_C(s) = \begin{cases} \bigcup_{s'\in C} V^*(s') & \text{if } s \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{10}$$

Define $\hat{V}_j(s) = \bigcup_{C\in\mathcal{C}_j} \theta_C(s)$. Similarly, we approximate any other correspondence from S to $\mathbb{R}$.

Theorem 2 *Under Assumption 1, we have* $\forall s \in S$*:*

(i) $V^*(s) \subset \ldots \subset \hat{V}_{j+1}(s) \subset \hat{V}_j(s) \subset \ldots \subset \hat{V}_1(s)$,

(ii) $\lim_{j\to\infty} d_H(\hat{V}_j(s), V^*(s)) = 0$,

(iii) *let* $\hat{W}_{t,j}(\cdot)$ *be a* j*-th piecewise constant approximation of* $W_t(\cdot)$*. Then,*

$$(\forall s \in S) \quad \lim_{t,j\to\infty} d_H\left(\hat{W}_{t,j}(s), V^*(s)\right) = 0. \tag{11}$$

Proof of theorem 2 Proof of (i) is obvious. Proof of (ii): We use Theorem 1 in [17] to show that $\hat{V}^j(s) \to V^*(s)$ in the Hausdorff distance. We need to show that $V^*(\cdot)$ is u.s.c. and compact valued correspondence. By the closed graph theorem (e.g., Theorem 17.11 in [3]), we only need to show that V^* has a closed graph.

Let $s_n \to s$, $x_n \to x$ $(n \to \infty)$ and $x_n \in V^*(s_n)$. Then, by Lemma 5, for all $n \in \mathbb{N}$, we have $w_n \in V^*$ such that $x_n = \Pi^\delta(c^{\beta\delta}_{w_n}(s_n), s_n, w_n)$. Take arbitrary $t \in \mathbb{N}$. Since $(w_n)_{n\in\mathbb{N}}$, where each $w_n \in W_t$, by Lemma 9, (w.l.o.g.) suppose we find $w \in W_t$ such that $w_n \to w$. By Lemma 3, $c^{\beta\delta}_{w_n}(s_n) \to c^{\beta\delta}_w(s)$ $(n \to \infty)$. Hence, by Assumption 1,

$$x = \lim_n x_n = \lim_n \Pi^\delta\left(c^{\beta\delta}_{w_n}(s_n), s_n, w_n\right) = \Pi^\delta\left(c^{\beta\delta}_w(s), s, w\right).$$

Therefore, $x \in W_t(s)$ for all $t \in \mathbb{N}$. From Lemma 9 $x \in V^*(s)$.

Proof of (iii) By Theorem 1, point (i), Theorem 1 in [17] and Lemma 9, we have:

$$V^*(s) = \bigcap_{t=1}^{\infty}\bigcap_{j=1}^{\infty} \hat{W}_{t,j}(s). \tag{12}$$

Applying Theorem 1 in [17], we obtain that each $\hat{W}_{t,j}(s)$ is a compact set. Let $s \in S$ be given. Let $(J_t)_{t\in\mathbb{N}}$ be some sequence satisfying the following conditions: If inclusion $W_{t+1}(s) \subset W_t(s)$ is strict, then J_t is chosen in such a way that:

$$W_{t+1}(s) \subset \hat{W}_{t+1,J_t}(s) \subset W_t(s).$$

If $W_t(s) = W_{t+1}(s)$, then $J_t = J_{t-1}$. Hence, $V^*(s) = \bigcap_{t=1}^{\infty} \hat{W}_{t+1,J_t}(s)$. By (12) we have $V^*(s) = \bigcap_{t=1}^{\infty} \hat{W}_{t+1,J_t}(s)$. Hence, we have $V^*(s) = \lim_{t\to\infty} \hat{W}_{t+1,J_t}(s)$ in the Hausdorff metric sense. As a result, for an arbitrary closed set G containing $V^*(s)$ we have $\hat{W}_{t,J_t}(s) \subset G$. Let t_0 be a number such that this inclusion is satisfied for all $t > t_0$. Since $j \to \hat{W}_{t,j}(s)$ is a descending family of sets, hence $\hat{W}_{t,j}(s) \subset G$, whenever $t > t_0$, and $j > J_{t_0}$. □

3.3 Discussion and Applications

Since it was introduced in [53], the problem of dynamic consistency has played an important role in many fields in economics. In particular, the problem of $\beta - \delta$ discounting has appeared in recent papers in such diverse topics as the theory of optimal consumption/savings [31], the role of liquidity constraints or commitment devices in dynamic models of self-control [38], design of dynamic, time-consistent environmental policies [35], temptation implications on the optimal taxation [36] or poverty traps [19].

Recently, in the accompanied paper, Balbus et al. [15] prove existence of the Markov-stationary NE of the quasi-hyperbolic discounting game in a class of bounded, Borel measurable functions and present a method to compute it. As compared to their paper, our assumptions and results differ among many dimensions. Firstly, they assume stronger conditions on a transition probability requiring a specific mixing form and an absorbing state. This allows them to weaken other conditions, most importantly dimensionality of the state space and work with the value functions that are not necessarily monotone. Their results offer a method to compute the extremal MSNE in a class of bounded, Borel measurable functions via a simple iterative scheme. This contrasts with the result presented in the current paper. First, here in Theorem 1, we prove existence of the MPNE in a class of monotone, Lipschitz continuous strategies that is not necessarily stationary, and second, Theorem 2 allows to compute the whole set of value functions generated by some Markovian NE in this class.

4 A Class of Stochastic Supermodular Games

In this section, we consider an N-player, discounted, infinite horizon, stochastic game in discrete time. The primitives of the class of games are given by the tuple $\left\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^N, Q, s_0\right\}$, where S is the state space, $A_i \subset \mathbb{R}^{k_i}$ player i action space with $A = \times_i A_i$, $\tilde{A}_i(s)$ the set of actions feasible for player i in state s, β_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, Q denotes a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of the next period state $s' \in S$, and finally $s_0 \in S$ is the initial state of the game. We assume that $S = [0, \bar{S}] \subset \mathbb{R}$ and that $\tilde{A}_i(s)$ is a compact Euclidean interval in $\mathbb{R}^{k_i}$ for each s, i.

Using this notation, a formal definition of a (Markov, stationary) strategy, payoff and a Nash equilibrium can be stated as follows. A set of all possible histories of player i till period t is denoted by H_i^t. An element $h_i^t \in H_i^t$ is of the form $h_i^t = (s_0, a_0, s_1, a_1, \ldots, a_{t-1}, s_t)$. A *strategy* for a player i is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where $\gamma_i^t : H_i^t \to A_i$ is a measurable mapping specifying an action to be taken at stage t as a function of history, such that $\gamma_i^t(h_i^t) \in \tilde{A}_i(s_t)$. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t, we have a Markov strategy given as $\gamma_i^t = \gamma_i$, then strategy Γ_i for player i is called a *Markov-stationary strategy*, and denoted simply by γ_i. For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_N)$, and initial state $s_0 \in S$, the expected payoff for player i can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t) dm_i^t(\Gamma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution (given by Ionescu–Tulcea's Theorem) induced on the space of all histories for Γ. A strategy profile $\Gamma^* = (\Gamma_i^*, \Gamma_{-i}^*)$ is a *Nash equilibrium* if and only if Γ^* is feasible, and for any i, and all feasible Γ_i, we have

$$U_i(\Gamma_i^*, \Gamma_{-i}^*, s_0) \geq U_i(\Gamma_i, \Gamma_{-i}^*, s_0).$$

4.1 Existence and Characterization

For our arguments in this section, we shall require the following assumptions.[10]

Assumption 2 We let:

[10] The assumption we impose here are very similar as those in the work of Amir [6] for $S \subset \mathbb{R}$.

- u_i be continuous on $S \times A$, and let u_i be bounded by 0 and $\bar{u}$,
- u_i be supermodular in a_i (for any s, a_{-i}), and have increasing differences in $(a_i; a_{-i}, s)$, and be increasing in (s, a_{-i}), (for each a_i),
- for all $s \in S$, the sets $\tilde{A}_i(s)$ be compact intervals and multifunction $\tilde{A}_i(\cdot)$ be upper hemicontinuous and ascending under both (i) set inclusion, i.e., if $s_1 \leq s_2$, then $\tilde{A}_i(s_1) \subseteq \tilde{A}_i(s_2)$, and (ii) Veinott's strong set order $\leq_v$ (i.e., $\tilde{A}_i(s_1) \leq_v \tilde{A}_i(s_2)$ if for all $a_{1i} \in \tilde{A}_i(s_1)$, $a_{2i} \in \tilde{A}_i(s_2)$, $a_{1i} \wedge a_{2i} \in \tilde{A}_i(s_1)$ and $a_{1i} \vee a_{2i} \in \tilde{A}_i(s_2)$),
- Q have a Feller property[11] on $S \times A$,
- $Q(s'|s, a)$ be stochastically supermodular in a_i (for any s, a_{-i}), have stochastically increasing differences in $(a_i; a_{-i}, s)$, and be stochastically increasing with a, s,
- $Q(\cdot|s, a)$ be a nonatomic measure for all $s \in S$, $a \in A$,

We now state some assumptions. Let V be the space of vectors of bounded, nondecreasing, u.s.c. value functions on S with values in $\mathbb{R}$:

$V := \{v : S \to \mathbb{R}_+^N,$ such that each v_i is nondecreasing, u.s.c. and each v_i is bouded by 0 and $\bar{u}\}$.

Endow V with the weak topology. See [29] for a proof that is a compact and metrizable topology on V. Define an auxiliary (or, super) one-period N-player game $G_w^s = (\{1, \ldots, N\}, \{\tilde{A}_i(s), \Pi_i\}_{i=1}^N)$, where payoffs depend on a weighted average of (i) the current within-period payoffs, and (ii) a vector of expected continuation values $w \in V$, with weights given by a discount factor:

$$\Pi_i(w_i, s, a) := (1 - \beta_i)u_i(s, a) + \beta_i \int_S w_i(s')Q(ds'|s, a),$$

where $w = (w_1, w_2, \ldots, w_N)$, $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_N)$, and the state $s \in S$. From now on by w_i, we will denote a typical element of w (similarly for v_i and v). As $w \in V$ is a vector of nondecreasing functions, under our assumptions, G_w^s is a supermodular game. Therefore, G_w^s has a nonempty complete lattice of pure strategy Nash equilibria (e.g., see [55]). By $NE(w, s)$ denote the set of Nash equilibria of game G_w^s restricted to nondecreasing functions on S (so hence measurable).

By $\mathcal{V}$ denote the set of all subsets of V partially ordered by the set inclusion. Having that, for any subset of functions $W \in \mathcal{V}$, define an operator B to be:

$$B(W) = \bigcup_{w \in W} \big\{ v \in V : (\forall s \in S)\, v(s) = \Pi(w, s, a^*(s)), \text{ for some } a^* \text{ s.t. } a^*(s) \in NE(w, s) \big\}.$$

We denote by $V^* \in \mathcal{V}$ the set of equilibrium values corresponding to all monotone, Markovian equilibria of our stochastic game.

Lemma 10 *Assume 2 and let $v^n \in V$, $a^n(s) \in A(s)$ for all $n \in \mathbb{N}$ and $s \in S$. If $v^n \to v^*$ and $a^n \to a^*$ pointwise in s, then $\Pi(v^n, s, a^n) \to \Pi(v^*, s, a^*)$.*

Proof of lemma 10 By Assumption 2, u_i is continuous in a. Observe that

$$(v_i, a) \to \int_S v_i(s')Q(ds'|s, a)$$

is continuous by Theorem 5.5. in [20]. Thus, Π is continuous in (v, a). □

[11] That is $\int_S f(s')Q(ds'|\cdot, \cdot)$ is continuous, whenever f is continuous and bounded.

Lemma 11 *Assume 2 and let $(v^n)_{n\in\mathbb{N}}$, where each $v^n \in V$ and $v^* \in V$. Let $v^n(\cdot) \to v^*(\cdot)$ pointwise and $a^n(s) \in NE(v^n, s)$ for all $s \in S$. Then, if $a^n \to a^*$ pointwise, then $a^*(s) \in NE(v^*, s)$ for all $s \in S$.*

Proof of lemma 11 By Assumption 2, we have continuity of u_i in a; hence, for all $a_i \in A_i(s)$, and $s \in S$ it holds that:

$$\Pi_i(v^n, s, a^n(s)) \geq \Pi_i(v^n, s, a^n_{-i}(s), a_i).$$

Taking a limit with $n \to \infty$, by Lemma 10, we obtain this inequality for a^* and v^*, hence $a^*(s) \in NE(v^*, s)$. □

Lemma 12 *Assume 2, then B maps $\mathcal{V}$ into itself. Moreover, $B(W) \neq \emptyset$, whenever $W \neq \emptyset$.*

Proof of lemma 12 Assume that $W \neq \emptyset$. Let $v \in W$. Then, v is (componentwise order) nondecreasing, and hence by Assumption 2, G^s_v is a supermodular game with parameter s. Hence, and by Milgrom and Roberts [43], there exist the greatest and the least selections and both are (componentwise order) nondecreasing in s. Again by Assumption 2, the extremal equilibria payoffs are both (componentwise order) nondecreasing in s. Let $\overline{w}$, $\underline{w}$ be (vectors) of such (extremal equilibrium payoff) functions. Thus, $\overline{w} \in B(W)$ and $\underline{w} \in B(W)$. □

Lemma 13 *Assume 2 and let W be a sequentially compact subset of V. Then, $B(W)$ is sequentially compact as well.*

Proof of lemma 13 Since $B(W) \in \mathcal{V}$ and, by Lemma 16 in the Appendix V, is sequentially compact set in the product topology which is Hausdorff, we just need to show $B(W)$ is sequentially compact. Let $(w_n)_{n\in\mathbb{N}}$, where each $w_n \in B(W)$ and suppose $w_n \to w$ pointwise. Let $(v_n)_{n\in\mathbb{N}}$ where each $v_n \in W$ and $(a_n(\cdot))_{n\in\mathbb{N}}$ be a sequence such that $w_n(s) = \Pi(v_n, s, a_n(s))$. By Lemma 16 in the Appendix, without loss of generality suppose $v_n \to v$ pointwise. Since W is sequentially compact, hence $v \in W$. Put D_w as a set of discontinuity points of w. Clearly D_w is at most countable. As Q is nonatomic, by Lemma 11, for each $s \in S$ there exists $a^*(s) \in NE(v, s)$ such that $w(s) = \Pi(v, s, a^*(s))$. Hence, $w \in B(W)$. □

Lemma 14 *Assume 2. If $W \subset B(W)$, then $B(W) \subset V^*$.*

Proof of lemma 14 Let $w \in B(W)$. Then, we have $v_0(\cdot) := w(\cdot)$ where $w(s) = \Pi(v_1, s, \gamma^1(s))$ for some $v_1 \in W$, Nash equilibrium $\gamma^1(s) \in NE(v_1, s)$ and all $s \in S$.

Then, since $v_1 \in W$ by the assumption, $v_1 \in B(W)$. Consequently, for $v_t \in W \subset B(W)$ $(t \geq 1)$ we can choose $v_{t+1} \in W$ such that $v_t(\cdot) = \Pi(v_{t+1}, \cdot, \gamma^{t+1}(\cdot))$ and $\gamma^t(\cdot) \in NE(v_{t+1}, \cdot)$. Clearly, the Markovian strategy γ generates payoff vector w. We next need to show this is a Nash equilibrium in the stochastic game for $s \in S$. Suppose that only player i uses some other strategy $\tilde{\gamma}_i$. Then, for all t and $s \in D_t$, we have $v^i_t(s) = \Pi_i(v_{t+1}, s, \gamma^t(s)) \geq \Pi_i(v_{t+1}, s, \gamma^{t+1}_{-i}(s), \tilde{\gamma}^{t+1}_i)$. If we take a T-th truncation $\gamma^{T,\infty} = \left(\left(\tilde{\gamma}^1_i, \gamma^1_{-i}\right), \ldots, \left(\tilde{\gamma}^T_i, \gamma^T_{-i}\right), \gamma^{T+1}, \gamma^{T+2}, \ldots\right)$, this strategy[12] cannot improve a payoff for player i. Indeed:

$$U_i(\gamma, s) \geq U_i(\gamma_{-i}, \gamma^{T,\infty}_i, s) \to U_i(\gamma_{-i}, \tilde{\gamma}_i, s)$$

as $T \to \infty$. This convergence has been obtained as u_i is bounded, and the residuum of the sum $U_i(\gamma_{-i}, \gamma^{T,\infty}_i, s)$ depending on $\left(\gamma^{T+1}, \gamma^{T+2}, \ldots\right)$ can be obtained as an expression bounded by $\bar{u}$, multiplied by β^T_i. Hence, $w(s)$ is a Nash equilibrium payoff for $s \in S$. Thus $B(W) \subset V^*$ □

[12] That is, player i uses strategy $\tilde{\gamma}$ up to period T and γ after that. Other players use γ.

We are now ready to summarize these results in the next theorem.

Theorem 3 *Assume 2. Then:*

(i) $\bigcap_{t=1}^{\infty} B^t(V) \neq \emptyset$,

(ii) $\bigcap_{t=1}^{\infty} B^t(V)$ *is the greatest fixed point of* B,

(iii) $\bigcap_{t=1}^{\infty} B^t(V) = V^*$.

Proof of theorem 3 We prove (i) and (ii). As $\mathcal{V}$ is a complete lattice, B is nondecreasing, by Tarski Theorem, B has the greatest fixed point, say W^*. Moreover, as B is nondecreasing, $\{W_t\}_{t=0}^{\infty}$, where $W_t = B^t(V)$, is a descending sequence (under set inclusion). We need to show that $\bigcap_{t=1}^{\infty} W_t = W^*$. Clearly, $\bigcap_{t=1}^{\infty} W_t \subset W_k$ for all $k \in \mathbb{N}$; hence

$$B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_{t+1} = \bigcap_{t=1}^{\infty} W_t.$$

To show equality, it suffices to show $\bigcap_{t=1}^{\infty} W_t \subset B(\bigcap_{t=1}^{\infty} W_t)$. Let $w \in \bigcap_{t=1}^{\infty} W_t$. Then, $w \in W_t$ for all t. By the definition of W_t and B, we obtain existence of the sequence $v^t \in W_t$ and Nash equilibria a^t such that

$$w(s) = \Pi\left(v^t, s, a^t(s)\right).$$

for all t and $s \in S$.

Since V is sequentially compact, without loss of generality, assume v^t converges to v^*. Moreover, $v^* \in \bigcap_{t=1}^{\infty} W_t$, since W_t is a descending family of sequentially compact sets in the product topology. Fix arbitrary $s \in S$. Without loss of generality, let $a^t \to a^*$, where a^* is some point from A. By Lemma 11, a^* is a Nash equilibrium in the static game $\Gamma(v^*, s)$.

We obtain $w \in B(\bigcap_{t=1}^{\infty} W_t)$. Hence, $\bigcap_{t=1}^{\infty} W_t$ is a fixed point of B, and, by definition $\bigcap_{t=1}^{\infty} W_t \subset W^*$.

To finish the proof, we simply need to show $W^* \subset \bigcap_{t=1}^{\infty} W_t$. Since $W^* \subset V$, $W^* = B(W^*) \subset B(V) = W_1$. By induction, we have $W^* \subset W_t$ for all t; hence, $W^* \subset \bigcap_{t=1}^{\infty} W_t$. Therefore, $W^* = \bigcap_{t=1}^{\infty} W_t$.

We prove (iii). First show that V^* is a fixed point of operator B. Clearly $B(V^*) \subset V^*$. So we just need to show the reverse inclusion. Let $v \in V^*$ and $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a profile supporting v. By Assumption 2, $\gamma_{2,\infty} = (\gamma_2, \gamma_3, \ldots)$ must be a Nash equilibrium almost everywhere (i.e., a set of initial states S_0 fir which $\gamma_{2,\infty}$ is not a Markov equilibrium must have a measure zero. Define a new profile $\tilde{\gamma}(s) = \gamma_{2,\infty}$ for $s \notin S_0$ and $\tilde{\gamma}(s) = \gamma$ if $s \in S_0$. Let $\tilde{v}$ be an equilibrium payoff generated by $\tilde{\gamma}$. Clearly, $\tilde{v} \in V^*$ is measurable and also $v(s) = \Pi(\tilde{v}, s, \gamma_1)$. Thus $v \in B(V^*)$ and hence $V^* \subset B(V^*)$. As a result, $B(V^*) = V^*$.

Finally, by definition (the greatest fixed point) of W^*, we conclude that $V^* \subset W^*$. To obtain the reverse inclusion, we apply Lemma 14. Indeed, $W^* \subset B(W^*)$, and, therefore, $W^* \subset V^*$ and we obtain that $V^* = W^*$.

Finally, observe that by the previous steps $V^* = \bigcap_{t=1}^{\infty} W_t$ and W_t is the set inclusion descending sequence, and by Lemma 13, all sets W_t are sequentially compact. Hence, by Lemma 17 in the Appendix, $V^* \neq \emptyset$. □

4.2 The Numerical Technique

Similarly, as in case of hyperbolic discounting model, we propose an approximation of V^* by a piecewise constant correspondence $\hat{W}_{t,j}$.

Theorem 4 *Under Assumption 2 for each $s \in S$, we have:*

(i) $V^*(s) \subset ... \subset \hat{V}_{j+1}(s) \subset \hat{V}_j(s) \subset \ldots \subset \hat{V}_1(s)$,

(ii) $\lim_{j\to\infty} d_H(\hat{V}_j(s), V^*(s)) = 0$,

(iii) *Let $\hat{W}_{t,j}(\cdot)$ be a j-th piecewise approximation of $W_t(\cdot)$. Then,*

$$\lim_{t,j\to\infty} d_H\left(\hat{W}_{t,j}(s), V^*(s)\right) = 0. \tag{13}$$

Proof of theorem 4 Proof of (i) is obvious. Proof of (ii): We apply Theorem 1 in [17]. To finish, we need to show that $V^*(s)$ has a closed graph.

Let $s_n \to s$ $(n \to \infty)$, $x_n \in V^*(s_n)$ for all n and $x_n \to x$. By Theorem 3 and Lemma 9, $x_n \in W_t(s_n)$ for all $t \in \mathbb{N}$. Then, there exists a sequence $(v_t^n)_{n\in\mathbb{N}}$, such that $x_n = v_t^n(s_n)$ with $v_t^n \in W_t$. Consequently, there exists $(v_\tau^n)_{\tau=1}^{t-1} \in \prod_{\tau=1}^{t-1} W_\tau$ such that $(\forall_{s\in S})$ $v_\tau^n(s) = \Pi(v_{\tau-1}^n, s, a_\tau^n)$ for some $a_\tau^n(s) \in NE(v_{\tau-1}^n, s)$. Without loss of generality, (using Lemma 16 from the Appendix) suppose, if we take a limit $n \to \infty$, then we obtain $v_\tau^n \to v_\tau$ pointwise in s for all $\tau \leq t$. Using Lemmas 16 and 13 in the Appendix, we obtain sequential compactness of all W_τ and $v_\tau \in W_\tau$. By Lemmas 10 and 11, we have $(\forall_{s\in S})$ $v_\tau(s) = \Pi(v_{\tau-1}, s, a_\tau(s))$ for some selection $a_\tau(s) \in NE(v_{\tau-1}, s)$. Applying again Lemma 11, we have then $x = \lim_n x_n = \Pi(v^{t-1}, s, a^t(s))$ for some selection $a^t(s) \in NE(v^{t-1}, s)$. Hence, $x \in W_t(s)$. Since $t \in \mathbb{N}$ is arbitrary, hence and by Lemma 9, $x \in V^*(s)$.

Proof of (iii). Observe that all W_t are sequentially compact. Hence, all $W_t(s)$ are compact sets for all s. The remainder of this proof is similar to the proof of part (iii) of Theorem 2. □

4.3 Discussion and Example

Theorem 3 establishes among others that the stochastic game has a (possibly nonstationary) Markov Nash Equilibrium in monotone strategies on the minimal state space of current state variables. Observe that conditions to establish that fact are weaker than the one imposed in [26]. Specifically, we do not require smoothness of the primitives nor any diagonal dominance conditions that assure that the auxiliary game has a unique Nash equilibrium, that is moreover continuous with the continuation value. Also the transition Q does not need to take a specific form like the one imposed in [14] with an absorbing state at 0. On the other hand, in [14] the authors do not require assumptions implying that the game has an nondecreasing equilibrium strategy and nondecreasing equilibrium value as imposed in this paper.

Moreover, Theorem 3 together with Theorem 4 offers a constructive method to characterize equilibrium value set and compute it in the Hausdorff distance using rigorous numerical technique.

In the accompanied paper [14], under some mixing assumptions on the stochastic transition, the authors are able to exploit the complementarity structure of this class of games and develop results for iterative procedures on the best response operator T to compute *both* values and pure strategies. This brings few interesting points. First, under assumptions of both papers (and this can be done indeed), one is able to show how to compute both the greatest and the least stationary MPNE values w^* and v^*, as well as the associated extremal pure strategy equilibrium. Of course, they are also Markov NE and $w^*, v^* \in V^* = B(V^*)$. Next, we can do such approximation for every iteration of our operators: $W_t \subseteq [v_t, w_t]$. So in such case, our iterations from the other paper provide interval bounds on the iterations on our strategic dynamic programming operator. Further, we conclude that $V^* \subseteq [v^*, w^*]$. That is, the set of value functions that are associated with MPNE belongs to an ordered interval between least and greatest MSNE. So although we cannot show that W_t (for $t \geq 2$) or V^* are ordered intervals of functions, we can use their iterative methods to calculate the two bounds using direct techniques of the accompanied paper.

This observation leads us to an important point linking our direct methods with MP/APS approach. Namely, in [2], they show that under certain assumption any value from $V^*(s)$ can be obtained in a bang-bang equilibrium of a *repeated* game with *imperfect monitoring*, i.e., one using extremal values from $V^*(s)$. Direct and constructive methods of Balbus et al. [14] can be hence used to compute two of such extremal values that support equilibrium punishment schemes that actually implement MPNE. This greatly sharpens the method by which we support all MPNE in our collection of dynamic games.

The stochastic, supermodular game can be applied to study price competition with durable goods, dynamic search with learning or symmetric equilibria in public goods games, among others. We finish this subsection with an example showing, how the proposed method can be used to study MPNE of the time-consistent public policy.

Consider a time-consistent policy game as analyzed in [52], for example. A (stochastic) game is played between a large number of identical households and the government. We study equilibria that treat each household identically. For any capital level $k \in S$ households choose consumption c and investment i treating level of the government spending G as given. The only way to consume tomorrow is to invest in the stochastic technology Q. The within-period preferences for the households are given by $u(c)$ (i.e., household does not obtain utility from public spending G). The government raises revenue by levying flat tax $\tau \in [0, 1]$ on capital income, to finance its public spending G. Each period the government budget is balanced, and its within-period preferences are given by: $u(c) + J(G)$. The consumption good production technology is given by constant return to scale function $f(k)$ with $f(0) = 0$ and $\frac{-kf''(k)}{f'(k)} \leq 1$. The timing of the game in each period is that the government and household choose their actions simultaneously.

We first assume that households and the government take price $R = f'(k)$ and profits $\pi = f(k) - f'(k)k$ as given. Assume that u, J, f are increasing, concave and twice continuously differentiable and Q satisfies Assumption 2. Each of the households then chooses investment i to solve:

$$\max_{i \in [0,(1-\tau)Rk]} (1-\beta)u((1-\tau)Rk + \pi - i) + \beta \int_S v_H(s)Q(ds|i).$$

By standard arguments, we see that the objective for the households is supermodular in i and has increasing differences in $(i; t, z)$, where $t = 1 - \tau$ and $z = (1-\tau)Rk + \pi = (1-\tau)f'(k)k + f(k) - f'(k)k$ (noting $-u''(\cdot) \geq 0$). Moreover, the objective is increasing in $t = 1 - \tau$ by monotonicity assumptions on u.

The government chooses t to solve:

$$\max_{t \in [\gamma,1]} (1-\beta)u(tRk + \pi - i) + (1-\beta)J(Rk(1-t)) + \beta \int_S (v_H(s) + v_G(s))Q(ds|i).$$

That is, the government maximizes the household utility as well as the additional utility that it obtains from public spending J, and its continuation v_G. Again, objective is supermodular in $1-\tau$ and has increasing differences in $(t=1-\tau, i)$ (as $-u''(\cdot)\geq 0$) and is monotone in (equilibrium) k. Moreover observe, although the objective is not nondecreasing in i, it is nondecreasing along extremal Nash equilibrium of the auxiliary game by the envelope theorem. To see that, by $(i^*, t^*)(v_H)$ denote an NE of the auxiliary game and observe that:

$$\frac{\partial}{\partial i}\Big[(1-\beta)u(t^*(v_H)Rk+\pi-i)+(1-\beta)J\left(Rk(1-t^*(v_H))\right)$$
$$+\beta\int_S (v_H(s)+v_G(s))Q(s|i)\Big]_{i=i^*(v_H)}$$
$$=\left[-(1-\beta)u'\left(t^*(v_H)Rk+\pi-i^*(v_H)\right)+\frac{\partial}{\partial i}\left[\beta\int_S v_H(s)Q(s|i)\right]_{i=i^*(v_H)}\right]$$
$$+\frac{\partial}{\partial i}\left[\beta\int_S v_G(s)Q(s|i)\right]_{i=i^*(v_H)}=\frac{\partial}{\partial i}\left[\beta\int_S v_G(s)Q(s|i)\right]_{i=i^*(v_H)}\geq 0.$$

Finally, $\gamma\in[0,1]$ is a parameter such that after tax income $(1-\tau(k))f'(k)k+f(k)-f'(k)k$ is monotone in k. Clearly, such number always exists. So, interestingly, the method developed in the paper can be extended and allows to study time-consistent public policies. That is, we are able to use our results to prove existence of MPNE, as well as compute the set of all MPNE values.

5 Concluding Remarks

In this paper, under mild conditions, we develop the strategic dynamic programming approach to a class of stochastic games and provide numerical method for computing the equilibrium value set that is associated with MPNE in the game. Few comments are now in order.

Observe that our procedure does not imply that the (Bellman type) equation $B(V^*)=V^*$ is satisfied for a *particular* value function v^*; rather, only by a *set* of value functions V^*. Hence, generally existence of a stationary Markov Nash equilibrium cannot be deduced using these arguments. Also, our method is in contrast to the original MP/APS method, where for any $w(s)$, one finds a continuation v; hence, the construction for that method becomes *pointwise*, as for any state $s\in S$, one can select a different continuation function v.

Our construction is related to the Cole and Kocherlakota [24] study of Markov-private information equilibria by the MP/APS-type procedure in function spaces. As compared to their study, ours treats different class of games (with general payoff functions, public signals and uncountable number of states) though. Also, recently and independently of our results, Doraszelski and Escobar [27] established a MP/APS-type procedure in function spaces for Markovian equilibria in repeated games with imperfect monitoring. Again their construction differs from ours as they require a finite number of actions, countable number of states and payoff irrelevant shocks.

Similarly, approximation results presented in this paper require few comments. Recall, our method approximates the set of equilibrium values on uncountable number of states. This is in contrast with all related papers that will be discussed in the moment.

In comparison with [34,51] method of MP/APS set approximation, our does not rely on *convexification* of the set of continuation values. Such step usually involves introducing sunspots or correlation devices into the game. Our method does not need any of this convexification or correlation.[13]

The argument presented in this paper share some properties of Chang [23] proposal of discretization technique for equilibrium value set approximation. This has been formalized by FMPA, who use Beer [17] result on approximation of correspondences by a step correspondence. Our approach is different as we approximate the set of functions, not a correspondence. Of course any set of function on a common domain can be represented as a correspondence, but doing such a step we loose important characterization of selections from this correspondence, however. In our approach, we select functions at the moment of defining an operator or every step of its approximation. This helps in computation of equilibrium strategies.

6 Appendix: auxiliary results

On the set of nondecreasing, real-valued functions in V let us introduce an equivalence relation:

$$\lfloor v \rfloor = \{w \in V, \text{ such that } v(s) \neq w(s) \text{ on at most countable set}\}.$$

We state few lemmas.

Lemma 15 *In each equivalence class $\lfloor v \rfloor$, there is exactly one u.s.c. function.*

Proof of lemma 15 First we show that each class possess at least one u.s.c. function. Let v be an arbitrary, nondecreasing function. Then, there are at most countably many discountinuity points $S_0 := \{s_1, s_2, \ldots, ...\}$. We define $\tilde{v}(s) := v(s)$ if $s \in S \setminus S_0$ and $\tilde{v}(s_j) = \limsup_{s \to s_j} v(s)$ if $s_j \in S_0$. Clearly, $\tilde{v}$ is a u.s.c. function and differs with v on at most countable set.

On the other hand, suppose there are two nondecreasing and u.s.c. functions w and v that differ on countable sets. Let s_0 be arbitrary point in which $w(s_0) \neq v(s_0)$. Then:

$$v(s_0) = \limsup_{s \to s_0} v(s) = \lim_{n \to \infty} v(s_n),$$

for some sequence[14] $s_n \to s_0$ and $s_n > s_0$. Without loss of generality, we can assume $v(s_n) = w(s_n)$ as the set of points in which v and w match is dense in S. Thus:

$$v(s_0) = \lim_{n \to \infty} v(s_n) = \lim_{n \to \infty} w(s_n) = w(s_0),$$

which contradicts $v(s_0) \neq w(s_0)$. □

Lemma 16 *Every sequence of nondecreasing, real-valued functions $(v_t)_t$ on S and bounded by a common value has a pointwise convergent subsequence.*

Proof of lemma 16 Let $v_t : S \to \mathbb{R}$, and $(v_t)_t$ be a sequence of nondecreasing functions bounded by some common value. By Lemma 15, there exist a sequence of functions $(\tilde{v}_t)_t$ such that each element is u.s.c. and differs from v_t on at most countable set, say D_t. Then (noting that the set V includes functions bounded by the common value), we can choose a

13 Also Cronshaw [25] proposes a Newton method for equilibrium value set approximation but cannot prove that his procedure converges to the greatest fixed point of our interest.

14 To avoid technical difficulties with defining the values of v at $\bar{S}$, we should extend the domain $[0, \bar{S}]$ to some $[0, \bar{S}']$ with $\bar{S}' > \bar{S}$. See [29] for a formal argument.

weakly convergent subsequence to some u.s.c. function $\tilde{v}$. Define $D := \bigcup_{t=1}^{\infty} D_t$. Clearly, it is a countable set, and for $S \backslash D$, we have $\tilde{v}_t(s) = v_t(s)$. On the other hand, the set of discontinuity points of v, say D_v, is also countable and on $s \in S \backslash (D \cup D_v)$ we have $v_t(s) = \tilde{v}_t(s) \to \tilde{v}(s)$ as $t \to \infty$. Since $D \cup D_v$ is countable, hence from a sequence v_t, we can choose a subsequence such that v_{t_n} is convergent on $D_v \cup D$ as well. As a result, $\lim_{n \to \infty} v_{t_n}(s)$ is this limit function. □

Lemma 17 *Let $X = \prod_{s \in S} K$ where K is a compact set in $\mathbb{R}^m$, and $(X, \mathcal{T})$ be a product topology. Let $(G_t)_t$ be a sequence of sequentially compact subsets of X. Assume $G_t(s) \neq \emptyset$ for all $s \in S$ and $(G_t)_t$ is descending in the set inclusion order. Then, $G^{\infty} := \bigcap_{t=1}^{\infty} G_t \neq \emptyset$.*

Proof of lemma 17 Let G be arbitrary sequentially compact subset of X. We show that $G(s) := \mathrm{Proj}_s(G)$ is compact for all K. As canonical projection on every s is continuous, as a function from X to K; hence, it is sequentially continuous. Hence, every set $G(s)$ is a compact subset of $\mathbb{R}$. Then, $\bigcap_{t \in \mathbb{N}} G_t(s) \neq \emptyset$ for all s and consequently:

$$\bigcap_{t \in \mathbb{N}} G_t = \prod_{s \in S} \bigcap_{t \in \mathbb{N}} G_t(s) \neq \emptyset.$$

□

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