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A constructive geometrical approach to the uniqueness of Markov stationary equilibrium in stochastic games of intergenerational altruism

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ABSTRACT

We provide sufficient conditions for existence and uniqueness of a monotone, Lipschitz continuous Markov stationary Nash equilibrium (MSNE) and characterize its associated Stationary Markov equilibrium in a class of intergenerational paternalistic altruism models with stochastic production. Our methods are constructive, and emphasize both order-theoretic and geometrical properties of nonlinear fixed point operators, and relate our results to the construction of globally stable numerical schemes that construct approximate Markov equilibrium in our models. Our results provide a new catalog of tools for the rigorous analysis of MSNE on minimal state spaces for OLG economies with stochastic production and limited commitment.

1. Introduction

Since the pioneering work of Phelps and Pollak (1968), Peleg and Yaari (1973), and Kydland and Prescott (1977, 1980), there has been great interest in studying dynamic economies without commitment. In some cases, the lack of commitment studied is between different generations of private agents, each who seek to develop enduring relations with successor generations that are needed to sustain coordinated action over time. This sort of intergenerational limited commitment problem arises in models of strategic altruistic growth, as well models where agents have preferences consistent with hyperbolic discounting, among others. In other cases, the commitment friction is between public policy agents and decisionmakers in the private economy who are trying to design mutually time-consistent equilibrium public policies. Examples of such situation is Ramsey equilibrium in models of optimal taxation, sustainable sovereign/public debt, and various monetary policy games.

In each of these models, in the end, the question of interest is the existence and characterization of the set of dynamic equilibria. As is well-known, in the presence of intertemporal commitment frictions, there are significant complications in

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verifying even the existence of subgame perfect equilibrium, let alone characterize the set of such equilibria. For reasons of tractability and numerical computation, researchers have more recently focused on the existence of pure strategy Markov stationary Nash equilibrium (MSNE) defined on ''minimal'' state spaces. Unfortunately, even when the set of subgame perfect equilibrium is nonempty, sufficient conditions that guarantee the existence of MSNE are not so clear. Further, in a great deal of recent applications of stochastic games, one seeks to compute elements of the set of MSNE (e.g., for calibration or estimation exercises). For such problems, new issues arise concerning the mathematical foundations of numerical procedures that are useful for such applied questions (as, in effect, much of this applied work implicitly assumes the existence of *unique* MSNE at each parameter value). Therefore, for the literature emphasizing the quantitative aspects of dynamic equilibria, perhaps the most important such issue is the stability of the set of MSNE in deep parameters. In this paper, we propose a new set of monotone iterative techniques that address all of these questions within the context of a well-studied class of models, namely stochastic overlapping generations models of growth with strategic bequests. In these economies, one assumes no commitment between successor generations, and object of interest is the set of MSNE. An important feature of our approach is to use properties of a stochastic transition structure. Specifically, under the conditions we present, we are able to obtain an Euler equation representation for MSNE that is both necessary and sufficient in all states. We are then able to use this Euler equation to show that *any* MSNE solution under our conditions must necessarily be the solution to a *decreasing*, continuous nonlinear operator that transforms an appropriate space of candidate equilibrium. Having this operator defined, we are able to provide sufficient conditions for MSNE existence in a compact subset of continuous functions. We then provide a sharp characterization of the order structure of the MSNE set (i.e., they are shown to form an ''antichain'').[1]

Next, and perhaps most strikingly, we provide a set of sufficient conditions under which *globally stable* iterative procedures are available for computing unique MSNE.[2] Moreover, our uniqueness result is valid relative to a very broad class of bounded measurable functions. Finally, we present explicit example where our uniqueness conditions do not hold and multiple MSNE exists. In this sense, we show our conditions for global stability are sharp. Our uniqueness result is particularly important as the class of economies for which it holds include parameterizations of stochastic OLG models found often in applied work.

Finally, we address issues related to the numerical approximation of MSNE in our economies. This question is important in applied work as many papers that seek to study dynamic economies without commitment must first numerically approximate MSNE (e.g., to estimate or calibrate the models to data). Therefore, we provide a catalog of theoretical results characterizing the properties of simple approximation schemes (e.g., discretization methods) that can be used to compute MSNE.

The remainder of the paper is organized as follows: Section 2 discusses how our methods and results fit into the existing literature. Section 3 defines the class of models we initially consider. Section 4 provides conditions under which our economies have (pure-strategy) MSNE, and under which the set of MSNE is a singleton. In Section 5, we provide extensions of our results based upon so-called ''mixed monotone'' operators, which allow us to obtain results for the nonseparable utility case. We also describe methods that construct approximate solutions for MSNE that achieve uniform error bounds relative to a simple discretization method (see Section 5.4 for example). Section 6 concludes with a discussion of applicability of our methods to other classes of stochastic games. At the end of the paper, we include an appendix that presents some definitions, a few abstract fixed point theorems that we use in the paper, as well as the proofs of all our results.[3]

2. Related literature

The environment we consider has long history in economics and dates back to the early work of Phelps and Pollak (1968) and Peleg and Yaari (1973).[4] The economy consists of a sequence of identical generations, each living one period, and deriving utility from its own consumption, as well as that of a successor generation. As agents cannot commit to plans, the ''dynastic family'' faces a time-consistency problem. In particular, each current generation has an incentive to deviate from a given sequence of bequests, consume a nonsustainable amount of current bequests, leaving little (or nothing) for subsequent generations.

Within this class of economies we study, conditions are known for the existence of semicontinuous MSNE, and have been established under quite general conditions (e.g. Leininger, 1986; Bernheim and Ray, 1987). An important step forward in characterizing the equilibrium was made in the work of Amir (1996b), where he introduces stochastic convex transitions structures into the game, and is able to establish the existence of MSNE in the space of continuous functions. This result has been further extended by Nowak and coauthors (e.g., see Nowak, 2006 or Balbus and Nowak, 2008). In this latter work, a key innovation was to introduce a class of stochastic transition structures that are assumed to be an

[1] We provide all the requisite definitions later in the paper when the results are presented.

[2] It bears mentioning obtaining sufficient conditions for the existence of *unique* MSNE has been an open question in the literature (e.g., see the discussion in Amir, 1996b).

[3] Apart from the proof of the equilibrium uniqueness result, which is included in the body of the paper.

[4] Versions of our model under perfect commitment have been also studied extensively in the literature, beginning with the important series of papers by Laitner (1979a,b), Loury (1981), and including more recent work of Alvarez (1999) or Laitner (2002) a.o.

(endogenous) mixtures of probability measures. It is important to note that in this body of work, when addressing the existence of MSNE, nonconstructive topological fixed point methods have been predominantly used, so characterizations of the set of MSNE have be primarily limited to compactness properties.[5]

Three important problems that have not been studied in existing literature concern: (i) constructive procedures[6] for studying MSNE, (ii) conditions for MSNE uniqueness, and (iii) methods for computing/approximating MSNE. Per the first and third question, a constructive method allows one to develop rigorous numerical procedures for applied work to build upon when computing MSNE. Per the second issue, conditions on uniqueness of MSNE allows one to study the stability of MSNE relative to deep parameters in a natural manner. Such stability results are often needed for a rigorous quantitative assessment of MSNE relative to data (e.g., in calibration or estimation methods).

To address all of these questions in our paper, we first provide sufficient conditions under which the set of pure-strategy MSNE forms an antichain in standard pointwise partial order. We then develop sufficient conditions under which the set of MSNE is actually a singleton. As our methods are constructive, these results provide a rich description of a class of iterative methods for computing pure-strategy MSNE. In particular, our methods provide sufficient conditions for globally stable approximate solutions relative to a unique non-trivial MSNE within a class of Lipschitz continuous MSNE.

Relative to the literature on recursive methods for dynamic economies, the key technical innovation in our approach is integration of order-theoretic, topological, and geometrical methods into a systematic study of the set of MSNE in a stochastic growth model without commitment. Although our techniques can be related to previous work on monotone methods (e.g., see Datta and Reffett, 2006 for a literature review and references), fixed point theory in ordered topological spaces (e.g., see Amann, 1977) and geometrical properties of mappings defined in abstract cones (e.g., see Krasnosel'skiĭ and Zabreĭko, 1984), what distinguishes the methods in this paper is our exclusive use of (iterative) fixed point theory for *decreasing* operators. To the best of our knowledge, this paper is the first application of iterative methods for decreasing operators for the study of Markov/recursive equilibrium in the economics literature.

The fact that the operator is decreasing, of course, greatly complicates matters. For example, unlike the increasing case studied in large body of work in macroeconomics stemming from monotone map approach first presented in Coleman (1991), as our operators are decreasing, they do not possess a fixed point property relative to complete partially ordered sets. To resolve the existence question, we integrate topological constructions (based on Schauder's theorem) into our order-theoretic geometric approach. A second complication of studying iterative methods for decreasing operators stems from the existence of ordered 2-cycles (or so-called "fixed edges"). To rule out such cycles, stability conditions for iterations (either global or local) require developing geometric conditions on fixed point operators (as opposed to, for example, simple order theoretic conditions). These conditions have typically not been required in previous work based upon increasing operators (e.g. Mirman et al., 2008). We show that such geometric conditions are available for our case, under reasonable conditions relative to applied work.

Finally, we can relate our methods to those in the existing literature that characterize subgame perfect or MSNE in dynamic economies without commitment. A "direct" approach to our class of problems has been undertaken by many authors. In this approach, existence of MSNE is obtained via fixed point methods in function spaces. This approach has a long line of important contributions,[7] including Leininger (1986), Bernheim and Ray (1987), Sundaram (1989), Curtat (1996), Amir (2002) and Nowak (2003, 2006, 2007). A second, albeit a less direct method of equilibrium construction, is the "promised utility method", best illustrated in the seminal work of Abreu et al. (1990), denoted by APS henceforth. This latter approach is based upon strategic dynamic programming arguments, and authors (e.g., Messner and Pavoni, 2004) seek to characterize the set of equilibrium values that are sustainable in a sequential equilibrium. In this method, the set of equilibrium values induced by sequential equilibria turns out to be the maximal fixed point of a monotone operator mapping between spaces of correspondences (ordered under set inclusion). Then, a dynamic equilibrium becomes a selection from the equilibrium correspondence (along with a corresponding set of sustainable pure strategies).

We should finally mention that although the promised utility approach has proven useful in some contexts,[8] in our class of models, it suffers from a number of well-known limitations. First, for our stochastic OLG models with strategic altruism, we do not need to impose discounting, which is typically required for promised utility methods. Second, the presence of noise over uncountable number of states introduces significant complications associated with the measurability of value correspondences that represent continuation structures. Equally as troubling, characterizations of stationary pure strategy equilibrium values (as well as implied pure strategies) are also difficult to obtain. Finally, it has not yet been shown by those that apply promised utility continuation methods how one can obtain any characterization of the long-run stochastic properties of stochastic games (i.e., equilibrium invariant distributions or ergodic sets).[9] All of these issues are resolved using our methods for the class of economies studied.

[5] With only few exceptions like Balbus and Nowak (2004).

[6] Which are important not only for theoretical issues such as equilibrium comparative statics, but also for characterizations of implied limiting distributions associated with pure strategies of the game.

[7] Also, see the work of Marcet and Marimon (1998) and Rustichini (1998) for a novel variation of this approach.

[8] See e.g. Atkeson (1991), Bernheim et al. (1999), Phelan and Stacchetti (2001), Judd et al. (2003), and Athey et al. (2005).

[9] For competitive economies, progress has been made. See Peralta-Alva and Santos (2010).

3. The model

We consider an infinite horizon dynastic production economy with limited commitment.[10] Time is discrete and indexed by $t=0,1,2,\ldots$. The economy has one-good each period and has access to a single store of value which is productive (hence, we refer to it as capital). Households are also endowed with a unit of time each period which they supply inelastically. A dynasty consists of a sequence of identical generations, each living one period, each caring about its successor generation. Any given generation divides its output x between current consumption c and investment $x-c$ for the successor generation. The current generation receives utility U from both its own current consumption and that of its immediate successor generation. Finally, there is stochastic production technology summarized by a mapping Q that transforms current stock and current investment into next period output.

We now provide some initial notation and formalities. Let $I(x)=[0,x]$ be the set of feasible choices of consumption for a generation with output x, $x \in I$ where the interval I is either bounded or unbounded,[11] I.e., $I=[0,S]$ with $S>0$ or $I=[0,\infty)$. By I^o denote interior of I. The preferences of the current generation are represented by a bounded, continuous (and, therefore, Borel measurable) utility function $U: I \times I \to \mathbb{R}_+$. The production technology is stochastic, and governed by probability distribution $Q(\cdot|x-c,x)$ of the next generation output parameterized by current investment $x-c$ and state x. Therefore, if the successor generation follows an integrable, stationary consumption policy $h: I \to I$, the expected payoff of the generation with endowment x and consuming $c \in I(x)$ is computed as follows:

$$\mathcal{W}(c,x,h) := \int_I U(c,h(y))Q(dy|x-c,x).$$

Assuming continuity of the problem's primitive data, by a standard application of Weierstrass's theorem, $\arg\max_{c\in I(x)} \mathcal{W}(c,x,h)$ exists for each $x \in I$, and can be viewed as a best response of a current generation to the policy h of its successor. A pure-strategy, Markov stationary Nash equilibrium is a measurable function h^* such that $h^*(x) \in \arg\max_{c\in I(x)} \mathcal{W}(c,x,h^*)$. Note, such an equilibrium remains an equilibrium if generations are allowed to use more general strategies.

4. Main results

We now present our main results. We start with a list of assumptions about preferences that are sufficient for the existence of MSNE. First, we state our assumptions on preferences.

Assumption 1 (*Preferences*). The utility satisfy:

- $U: I \times I \to \mathbb{R}_+$ is of the form $U(c_1,c_2)=u(c_1)+v(c_2)$, where u and v are strictly increasing on I. Moreover v is bounded, continuously differentiable on[12] I°, u is twice continuously differentiable on I°, strictly concave and continuous on I,
- $v(0)=0$.

We now state the following assumption on the stochastic transition probability Q.

Assumption 2 (*Technology*). Transition probability Q satisfies

- $Q(\cdot|x-c,x)=(1-g(x-c))\delta_0(\cdot)+g(x-c)\lambda(\cdot|x)$, where
- $g: I \to [0,1]$ is strictly increasing, concave on I and twice continuously differentiable on I°,
- $(\forall x \in I)\lambda(\cdot|x)$ is a Borel transition probability on I, moreover $\lambda(\cdot|x)$ satisfies a Feller Property, i.e., the function $x \to \int_I f(y)\lambda(dy|x)$ is continuous whenever f is continuous and bounded measurable, and $\forall(x \in I), \lambda(\{0\}|x)=0$,
- δ_0 is a probability measure concentrated at point zero.

Before preceding to our main existence theorems, we make a few remarks on these assumptions. We begin with a comparison of the technical aspects of our conditions versus the existing literature. First, as in Nowak (2006), we assume Q is a convex combination (or mixture) of two distributions, with λ stochastically dominating δ_0. Hence, the stochastic structure of our game places probability $1-g$ on the possibility that the next period capital is 0, and probability g it is drawn from λ (where these probabilities are endogenously determined in equilibrium). Second, notice all the nonconvexities of transition Q are given by function $1-g$. Now, as δ_0 is a Dirac delta measure concentrated at zero, and $v(h(0))=0$, essentially, all such production nonconvexities are negligible. Third, a comparison between our technologies and those in Amir (1996b) for the case of unbounded state space I can be made. On the one hand, Amir's transition Q is not dependent on the stock, rather it depends on investment only. This situation may lead to the existence of a unique, yet trivial, stationary Markovian equilibrium associated with MSNE. We seek to avoid this situation, and allow for more

[10] More precisely, we are studying a dynamic stochastic production economy with both capital and labor, but with preferences not defined over leisure. Therefore, each period, agents supply labor inelastically. See remarks below on the formal interpretation of our stochastic production function.

[11] To denote the latter case, we will sometimes write $S=\infty$.

[12] For any set A, A° is said to be an interior of A.

general stochastic transitions. On the other hand, Amir does not require the mixing specification for stochastic production that we impose (although, of course, he does allow it). He also does not require one of distributions to be Dirac delta.[13]

Next, from an economic perspective, our assumptions on Q generate a large class of stochastically monotone and stochastically concave transition probabilities (e.g., see Amir, 1996b for discussion). For example, one important simple example has $g(x-c)=\beta(x-c)^{\alpha}$ or $g(x-c)=\beta(1-e^{-(x-c)})$ where $1>\alpha>0$ and β are sufficiently small positive numbers. Such functions (Cobb–Douglas and CARA) have found extensive use in applied macroeconomic papers. We should mention that apart from Amir (1996b) and Nowak (2006), applied papers of Horst (2005) and examples in Curtat (1996) use similar transition specification given by a convex combination of two stochastically ordered distributions. Hence, the difference between ours and their approaches is that we take one of these distributions to be Dirac delta at zero.

With this discussion of assumptions in mind, we now proceed with the construction of the set of MSNE under our Assumptions 1 and 2. First, under these conditions, notice the objective function $\mathcal{W}(c,x,h)$ is strictly concave and continuous in c on $I(x)$. Hence, given any integrable and feasible continuation strategy h for the successor generation, there is a well defined (measurable) best response mapping/operator given by

$$A(h)(x)=\arg\max_{c\in I(x)}\mathcal{W}(c,x,h). \tag{1}$$

We now discuss existence of MSNE.

4.1. Existence

We first prove existence of MSNE within the class of Lipschitzian functions. To do this, consider a collection:

$$H=\{h\in\mathcal{C}(I)\,|\,(\forall y\in I)0\le h(y)\le y\},$$

where $\mathcal{C}(I)$ is the set of nonnegative, continuous functions from I into I. Endow the set H with the pointwise partial order and the topology of uniform convergence on compacta of I. As a matter of notation, we denote uniform convergence as "$\stackrel{u}{\to}$".[14] Denote the zero element of H by $\underline{\theta}$ (with $\underline{\theta}(x)=0$), and the identity map by $\overline{\theta}$ (where $\overline{\theta}(x)=x$).

We seek MSNE within a subset of H. Let $\mathcal{L}_{\mathcal{M}}$ be a set of all increasing and locally Lipschitz functions with common modulus on all compact subsets of I^o. Formally,

$$\mathcal{L}_{\mathcal{M}}:=\{h\in H:\forall_{W\in\mathcal{C}_0}0\le h(y_1)-h(y_2)\le\mathcal{M}(W)|y_1-y_2|,y_2\ge y_1\in W\},$$

where $\mathcal{C}_0$ denotes a set of compact subsets of $I^\circ=(0,\infty)$ and function $\mathcal{M}:\mathcal{C}_0\to\mathbb{R}_+$. An important special subclass of $\mathcal{L}_{\mathcal{M}}$ is the class of all increasing and Lipschitz continuous elements with common Lipschitz constant[15] M:

$$L_M=\{h\in H|(\forall y_1\le y_2\in I)0\le h(y_2)-h(y_1)\le M(y_2-y_1),h(0)=0\}.$$

Under Assumptions 1 and 2, for $h\in H$, the objective function for the current generation is given by

$$\mathcal{W}(c,x,h)=u(c)+g(x-c)\int_I v(h(y))\lambda(dy|x).$$

Further, as u and g are both differentiable on I°, after linearizing the objective $\mathcal{W}$, using the first order condition, we can define a mapping $\zeta(c,x,h)$ as follows:

$$\zeta(c,x,h):=u'(c)-g'(x-c)\int_I v(h(y))\lambda(dy|x). \tag{2}$$

We now define our fixed point operator implicitly using the Euler equation $\zeta(c,x,h)=0$. To do this, we first study the properties of the mapping ζ in each of its arguments. Fix (x,h), $x\in I,x>0$, $h\in H$. Then, given the concavity of value function $\mathcal{W}$ in c, for each (x,h), a necessary and sufficient condition for $c^*(x,h)\in I^\circ$ to be optimal is $\zeta(c^*(x,h),x,h)=0$. Further, if $(\forall c\in I)\zeta(c,x,h)>0$, then, the optimal $c^*=x$; while, if $(\forall c\in I)\zeta(c,x,h)<0$, the optimal $c^*=0$. We use the optimal policy function $c^*(x,h)$ to define a nonlinear operator A with values given pointwise by $Ah(x):=c^*(x,h)$. By the above reasoning, we have $A\underline{\theta}=\overline{\theta}$. Further, given that for all pairs (x,h), the Euler equation is necessary and sufficient for the optimal solutions $c^*(x,h)$, any MSNE must necessarily be a fixed point of our nonlinear operator.

To provide conditions where there exists some positive valued function $\mathcal{M}$ such that A maps $\mathcal{L}_{\mathcal{M}}$ into itself, we make the following final assumption.

Assumption 3. Assume that

- $(\forall c\in I\backslash\{0\})u''(c)<0$,
- for any $x\in I\backslash\{0\}$, any $c\in[0,x)$ and $g''(x-c)<0$ the value of $|g'(x-c)/g''(x-c)|$ is bounded (by some positive constant M_g),[16]

[13] See his Example 2.

[14] Note, if I is a bounded interval, our topology is equivalent to sup-norm topology. Also, our topology and notion of convergence is precisely that studied by Amir (1996b), and is often referred to as the compact-open topology.

[15] Note, if $M_0:=\sup\{\mathcal{M}(W):W\in\mathcal{C}_0\}<\infty$, then $\mathcal{L}_{\mathcal{M}}\subseteq L_{M_0}$. Also, if I is a closed interval (and, hence, compact), the set $\mathcal{L}_{\mathcal{M}}$ is equivalent to L_M for some M.

[16] Observe also that we allow $g''(0)=-\infty$.

- collection of measures $\{\lambda(\cdot|x), x \in I\}$ is stochastically decreasing[17] with x,
- the cdf of $\lambda(\cdot|x)$ say $F_\lambda(\cdot|x)$ is differentiable in x, i.e., for every $y \in I$ there exists derivative $(\partial/\partial x)F_\lambda(y|x)$,
- for each compact subset of I^o (say W) there exists a constant M_F^W such that $|(\partial/\partial x)F_\lambda(y|x)| \le M_F^W(1-F_\lambda(y|x))$. Let $M_F := \sup\{M_F^W : W \in \mathcal{C}^o\}$.

Define (for a given positive, integrable f) function $p_f : I \to \mathbb{R}_+$ with $p_f(x) := \int_I f(y)\lambda(dy|x)$. We start with a short lemma characterizing function p_f under Assumption 3.

Lemma 1. *Let Assumption 3 be satisfied. Then for all compact sets $W \subset I^o$, $x \in W$ and continuous, increasing function f the following holds*:

$$|p_f'(x)| \le M_F^W p_f(x).$$

The next lemma shows the existence of $a \overset{u}{\to}$ compact set $\mathcal{L}_\mathcal{M}$ such that A maps $\mathcal{L}_\mathcal{M}$ into itself under our general assumptions, as well as gives conditions under which A maps L_M into itself for some fixed number M.

Lemma 2. *Let Assumptions 1–3 be satisfied. Then*:

(i) *$A : \mathcal{L}_\mathcal{M} \to \mathcal{L}_\mathcal{M}$, for some function $\mathcal{M}$. Further, A is continuous and decreasing on $\mathcal{L}_\mathcal{M}$,*[18]
(ii) *assume additionally $M_F < \infty$, and let $M = 2 + M_F M_g$. Then, $A : L_M \to L_M$. Further, A is continuous and decreasing on L_M.*

We now provide our first important result on existence of MSNE. It is important to note that in this result, aside from proving the existence of Lipschitzian MSNE, we also characterize the order structure of the equilibrium set in L_M. In the theorem, Ψ_A denotes the set of fixed points of the operator A.

Theorem 3. *Assume 1–3. Let H_0 be a set from Lemma 2 then*:

(i) *$\Psi_A \subset \mathcal{L}_\mathcal{M}$ and is a non-empty anti-chain (i.e., has no ordered elements in $\mathcal{L}_\mathcal{M}$).*
(ii) *If additionally $M_F < \infty$, then, for a constant $M := 2 + M_F M_g$, the set of fixed points Ψ_A of A in L_M is a non-empty anti-chain.*

A few remarks on Assumption 3 and Theorem 3 are warranted.

First, requiring stochastic monotonicity of λ for the transition structure is stronger than assumptions used in Nowak (2006) to show existence of a monotone, continuous MSNE. However, to show existence of a *Lipschitz* continuous MSNE, Nowak (2006) assumes that λ does not depend on x.[19] Using our Euler equation approach, for λ constant, existence of MSNE in L_1 can be easily established. This case is not particularly interesting, however, as the continuation $\int_I v(y)\lambda(dy)$ becomes constant.[20] Given this, our theorem is somewhat different, as we show existence of a monotone, Lipschitz continuous MSNE with $\lambda(\cdot|x)$ dependent on x. For this result, relative to Nowak (2006), we require some additional assumptions bounding derivatives of g and p_f.

Second, to see how conditions on p_f in this theorem maybe satisfied, observe that if I is bounded, and for any $x \in I$, the measure $\lambda(\cdot|x)$ has a density, and the ratios of derivative (with respect to x) of each of these densities (and densities themselves) are bounded by some constant $\overline{\rho}$ (i.e., $|(\partial/\partial x)\rho(y|x)/\rho(y|x)| \le \overline{\rho}$ a.e.). Therefore, this assumption is indeed satisfied with $M_F < \infty$. So, clearly, verifying this condition in applications is direct.

Third, as our operator A embodies all the *necessary* conditions for any MSNE, exploiting the antitone structure of the operator A, we are able to sharpen the characterization of the set of MSNE relative to existing results. Namely, the set of MSNE forms an *antichain*. This fact could be important, for example, if one considers questions of the existence (or lack thereof) of monotone comparative statics on the deep parameters of the game (as such conditions will be very difficult to obtain in this class of games).

Finally, under the conditions of the theorem, generally the set of MSNE in L_M is not a singleton (see Example 3).[21] We will consider the uniqueness question later in this section, and this will require further restrictions on primitives.

We finish this section with an example of probability measures that satisfy Assumption 3.[22]

[17] We say that the collection of measures $\lambda(\cdot|x)$ is stochastically decreasing with x iff function $x \to \int_I f(y)\lambda(dy|x)$ is decreasing for any integrable, positive real valued and increasing f of I.

[18] I.e., $\forall h_2, h_1 \in \mathcal{L}_\mathcal{M}$ with $h_2 \ge h_1$, $Ah_1 \ge Ah_2$.

[19] See Amir (1996b) for a corresponding assumption that Q does not depend on x.

[20] Also MSNE uniqueness trivially follows from the monotonicity of the expected value function operator mapping reals to reals.

[21] One method for checking uniqueness of the fixed point of A is to show it is upward directed. The partially ordered set $(B, \le)$ is upward directed if and only if for any $b_1, b_2 \in B$ there exists $b_3 \in B$ such that $b_1 \le b_3$ and $b_2 \le b_3$ (Cid-Araújo, 2004).

[22] More examples concerning this and other results from the paper can be found in a working paper version (available from the authors upon request) of this manuscript.

Example 1. Let $I=[0,1]$ and $\rho(y|x)=(2-x)e^{-(2-x)y}/(1-e^{-(2-x)})$. The distribution function is

$$G_\rho(y|x)=\frac{1-e^{-(2-x)y}}{1-e^{-(2-x)}}.$$

Note that $G_\rho(y|\cdot)$ is a composition of the increasing function on $[0,1]$ $t\to(1-t^y)/(1-t)$ with increasing function $e^{-(2-x)}$; hence, distribution function F_ρ is increasing. This implies that λ is stochastically decreasing. Further, we can alternatively express the density as $\rho(y|x):=\exp(-(2-x)y+\Phi(x))$, where $\Phi(x):=\ln(2-x)-\ln(1-e^{x-2})$. Note that

$$|\Phi'(x)|\le\frac{1}{2-x}+\frac{e^{x-2}}{1-e^{x-2}}\le 1+\frac{e^{-1}}{1-e^{-1}}=\frac{1}{1-e^{-1}}.$$

Therefore, we have

$$\left|\frac{\partial\rho(y|x)}{\partial x}\right|\le\rho(y|x)(y+\Phi'(x)),$$

$$\le\rho(y|x)\left(1+\frac{1}{1-e^{-1}}\right)=\rho(y|x)\frac{2-e^{-1}}{1-e^{-1}},$$

$$(\partial/\partial x)|F_\lambda(y|x)|\le\int_I\chi_{[y,\infty]}(y')\left|\frac{\partial\rho(y'|x)}{\partial x}\right|dy'\le\frac{2-e^{-1}}{1-e^{-1}}F_\lambda(y|x).$$

Hence, Assumption 3 is satisfied.

4.2. Uniqueness

In this section, we turn to the question of the MSNE uniqueness. In particular, we prove the uniqueness of MSNE relative to a class of bounded measurable strategies. This fact shall be particularly important (as later in the paper, we study the properties of approximation procedures for constructing numerical solutions). It is equally as important if one wants to compute numerically equilibrium comparative statics (in the set of MSNE) in the deep parameters of the model.

We begin by letting $P=\{p:I\to\mathbb{R}_+|p \text{ is bounded and Borel measurable}\}$. Define an operator $B:P\to P$ as follows. For $p\in P$, compute

$$B(p)(x)=\int_I\nu(c_p^*(y))\lambda(dy|x),$$

with

$$c_p^*(x)=\arg\max_{c\in I(x)}\{u(c)+p(x)g(x-c)\}.$$

By Assumptions 1 and 2, we obtain

$$c_p^*(x)=\begin{cases}x & \text{if } u'(x)-p(x)g'(0)\ge 0,\\ c_0^p(x) & \text{if } u'(0)-p(x)g'(x)>0>u'(x)-p(x)g'(0),\\ 0 & \text{if } u'(0)-p(x)g'(x)\le 0,\end{cases}$$

where $c_0^p(x)$ is the c solving equation $u'(c)-p(x)g'(x-c)=0$.

Now, consider the following functional equation:

$$p(x)=\int_I\nu(c_p^*(y)))\lambda(dy|x). \tag{3}$$

This functional equation is easily shown to be well defined. Further, it is clear we have a solution to the functional equation (3) $p^*\in P$ if and only if $h^*(x)=c_{p^*}^*(x)$ is a MSNE.

The next theorem presents conditions under which B has a unique fixed point (and hence, MSNE is unique).

Theorem 4. *Let conditions 1 and 2 be satisfied with I bounded. Assume that $\lim_{c\to0^+}u'(c)=\infty$, $u''(c)<0$ for $c>0$, and $(\exists r,0<r<1)$ such that $(\forall x\in I,x>0)$ the following holds*:

$$(\forall c\in I^\circ(x))\quad\frac{c\nu'(c)}{\nu(c)}\le r\left[-\frac{cu''(c)}{u'(c)}-\frac{cg''(x-c)}{g'(x-c)}\right]. \tag{4}$$

Then, B is decreasing and has a unique fixed point p^* in P° such that

$$(\forall p_0\in P^\circ)\quad\lim_{n\to\infty}\|p_n-p^*\|=0, \tag{5}$$

where p_n is computed recursively as $(\forall n\geq 1)p_n = B(p_{n-1})$ for all p_0. Moreover, we have the following estimate of a convergence rate:

$$\|p_n - p^*\| \leq M_B(1-\tau^{r^n}), \tag{6}$$

where $M_B>0$ and $0<\tau<1$ are positive constants that depend on the initial choice of p_0.

Theorem 4 gives the sufficient conditions for the uniqueness of fixed points for the operator B. Our uniqueness result is robust to a *very large* space of functions (i.e., the space of bounded, real-valued, Borel measurable functions on I). So our unique fixed point corresponds to the unique measurable MSNE h^*. The theorem also provides uniform error bounds, and rates of convergence of iterations on B to this unique fixed point from any initial guess p_0 at the fixed point. If we add Assumption 3, and link our results with results of Theorem 3, this implies that this unique MSNE corresponds to existence in the space[23] L_M. Therefore, the results prove uniqueness relative to a very large set of functions (namely, the space of bounded measurable functions), and while we have existence in a very narrow set (Lipschitz continuous functions). So the result of the theorem is very strong.[24]

As Theorem 4 is the main contribution of our paper, we present the proof of this result to highlight the role of particular assumptions we invoke in its construction.

Proof of Theorem 4. The strategy of the proof is to show that conditions of Guo et al. (2004) theorem are satisfied (see Theorem 9 in Appendix). Note first that P is normal solid cone with natural product order and sup-norm, and P° is a set of elements from P with strictly positive infimum.

We now show that $B(p)\in P^\circ$ whenever $p\in P$. Begin by considering constant functions. Let $\omega(c,x,p) = u(c)+pg(x-c)$ and recall that $c_p(x)$ is decreasing in p and increasing and continuous in x. Hence[25] by Assumption 2 $B(p)(x) = \int_I v(c_p^*(y))\lambda(dy|x)$ is continuous in x. On the other hand observe that

$$\lim_{c\to 0^+}\frac{\omega(c,x,p)}{\partial c} = \lim_{c\to 0^+}(u'(c)-pg'(x-c)) = \infty.$$

Hence by definition of c_p we obtain that $c_p^*(y)>0$ for $y>0$. As v is strictly positive on $I\backslash\{0\}$, and $\lambda(\cdot|x)$ is not Dirac delta at 0, this implies that $B(p)(x) := \int_I v(c_p^*(y))\lambda(dy|x)>0$, for all $x\in I$. Together with the continuity of $B(p)(\cdot)$, we have $B(p)\in P^\circ$ whenever p is constant. Now, consider arbitrary $p\in P$ and denote $\overline{p} := \sup_{x\in I}p(x)$. By the same reasoning as before, we conclude that B is decreasing in p. Therefore, $\inf_{x\in I}B(p)(x)\geq \inf_{x\in I}B(\overline{p})(x)>0$. Hence $B(p)\in P^\circ$ for all $p\in P^\circ$.

Next, for a given $r, 1>r>0$, consider a function $\phi_r:[0,1]\to\mathbb{R}_+$, $\phi_r(t) = t^r B(tp)(x)$. We will show that ϕ_r is increasing with t on $(0,1)$. Adding continuity of ϕ_r from the left at 1, we will conclude that $\exists r, 0<r<1$ such that $\phi_r(t)\leq\phi_r(1)$; hence, $t^r B(tp)\leq Bp$ as required by Guo et al. (2004) theorem.

Recall $p\to c_p^*(x)$ is decreasing and continuous in the topology $\xrightarrow{u}$. By the same argument $t\to c_{tp}^*$ is decreasing and continuous. By definition of B it is sufficient to show that the function $t\to t^r v(c_{tp}^*(y))$ is increasing for all $y\in I$. Clearly $c_{tp}^*(0)=0$ for all $t\in(0,1]$. For arbitrary $y\in I^\circ$, let us divide interval $T^0 := (0,1]$ into two disjoints parts $T^0 = T_1^y\cup T_2^y$, where $T_1^y := \{t\in T^0 : c_{tp}^*(y)\in(0,1)\}$ and $T_2^y := \{t\in T^0 : c_{tp}(y) = y\}$. Note that T_1^y is open and T_2^y is closed. Since $t\to t^r v(c_{tp}^*(y))$ is continuous, we just need to show that this function is increasing in all T_i^y. It is easy to see that this function is increasing on T_2^y. Let $t\in T_2^y$, and $c(t) := c_{tp}^*(y)$. Clearly $c(t)$ solves an equation $(\partial/\partial c)\omega(c,y,tp)=0$. Since $u''(c)<0$ for $c>0$, by Implicit Function Theorem we obtain that $c'(t) := (d/dt)c_{tp}^*(y)$ exists and

$$c'(t) = \frac{p(y)g'(y-c(t))}{u''(c(t))+tp(y)g''(y-c(t))}. \tag{7}$$

Further

$$\frac{d}{dt}(t^r v(c(t)) = rt^{r-1}v(c(t))+t^r v'(c(t))c'(t) = t^{r-1}v(c(t))\left(r+t\frac{v'(c(t))}{v(c(t))}c'(t)\right).$$

By Eq. (7):

$$t\frac{v'(c(t))}{v(c(t))}c'(t) = t\frac{v'(c(t))}{v(c(t))}\frac{p(y)g'(y-c(t))}{u''(c(t))+tp(y)g''(y-c(t))} = \frac{\dfrac{v'(c(t))}{v(c(t))}}{\dfrac{u''(c(t))}{tp(y)g'(y-c(t))}+\dfrac{g''(y-c(t))}{g'(y-c(t))}} = \frac{\dfrac{v'(c(t))}{v(c(t))}}{\dfrac{u''(c(t))}{u'(c(t))}+\dfrac{g''(y-c(t))}{g'(y-c(t))}} = -\frac{\dfrac{v'(c(t))}{v(c(t))}}{-\dfrac{u''(c(t))}{u'(c(t))}-\dfrac{g''(y-c(t))}{g'(y-c(t))}}\geq -r, \tag{8}$$

[23] Example 2 shows that it is indeed possible for a common functional forms of preferences and production to satisfy all these assumptions.

[24] When solving functional equations, solutions exist in subclasses of function. For a strong uniqueness result, one seeks existence in a narrow class, and uniqueness in a broad class. It bears mentioning, for our class of economies, we cannot imagine requiring a uniqueness argument in a larger class of functions than bounded measurable functions (hence, our uniqueness result is a strong as seems possible for our class of games).

[25] Observe that the proof of this theorem remains valid if we change assumption on Feller property of $\lambda(\cdot|x)$ into first order stochastically decreasing assumption.

where the last inequality follows from (4). Combining (7) and (8) we obtain that $(d/dt)(t^r v(c(t)) \geq 0$, and therefore $t^r v(c(t)) = t^r v(c^*_{tp}(y))$ is increasing on the interval T_2^y. Hence, we obtain monotonicity of $\phi_r(\cdot)$ on the whole interval [0,1].

As a result, we have that $t^r B(tp) \leq Bp$ for any $t, 0<t<1$, any $p \in P^\circ$ as in Theorem 9. Therefore, we conclude that B has a unique fixed point in P°, and conditions (5) and (6) hold. □

The geometric intuition behind the proof of Theorem 4 is as follows: since the operator B is decreasing, it may have multiple, unordered fixed points (i.e., our existence theorem show that set of fixed points is nonempty antichain). The conditions in Theorem 2 assert, however, that the operator B operator is "e-convex" (see Guo and Lakshmikantham, 1988) for a discussion). In particular, it is contraction along cone origin rays. This is a very strong infinite dimensional geometric condition for an operator to satisfy, and proves sufficient for existence of a *unique* fixed point.

From an economic perspective, the condition (4) has a simple interpretation in terms of elasticities of payoffs. In particular, the condition on the primitive data of the model used to generate current period returns requires that the sum of their elasticities in absolute value (namely, the elasticities implied by the derivatives of u' and g' with respect to c) exceed the elasticity of consumption for continuation utility (which is an integral parameterized by v). That is, the percentage change in continuation dynastic utility v resulting from a percentage change in c cannot be *too high*.[26]

Although, of course, this condition (4) is restrictive (and, indeed, one cannot expect MSNE uniqueness under general conditions in this class of games), it is still satisfied for many utility functions including those often used in applications. To understand the nature of the restrictions, let use provide a simple illustration of the role of the conditions.

Example 2. Let I be bounded. Consider the time separable power utility function: $U(c_1,c_2) = c_1^\alpha + \delta c_2^\beta$, where $0<\delta \leq 1$ and $\alpha>0, \beta>0$ with $\alpha+\beta<1$. Observe that U satisfies Assumption 1 and conditions in Theorem 4, whenever stochastic production parameterized by the function g satisfies Assumption 2. To see this, follow the inequalities for c given as in (4):

$$\frac{cv'(c)}{v(c)} = \beta < 1-\alpha \leq 1-\alpha-\frac{cg''(x-c)}{g'(x-c)} = -\frac{cu''(c)}{u'(c)} - \frac{cg''(x-c)}{g'(x-c)},$$

where the first inequality is satisfied by assumption, and the second follows from the strict monotonicity and concavity of g. Since, the inequality is strict, $\exists r$ with $0<r<1$, such that condition (4) holds for x in bounded I.

So, for example, under many standard power/CRRA utility specifications used in applied macroeconomic modeling, our condition is satisfied. Further, to understand the nature of our condition, notice that by dividing inequality (4) by c, for continuous v with $v(0)=0$, the left hand side of our inequality tends to infinity with $c \to 0$. So, essentially for our condition to hold for c close to 0, one needs the absolute risk aversion measure $-u''/u'$ be unbounded, and also tend to infinity with c limiting to 0. Hence, condition (4) is not satisfied for utility functions such as $u(c) = \ln(c+1)$ or CARA preferences given e.g. by $u(c) = 1-e^{-c}$. Finally, observe that Example 2 suggests that for CRRA utilities, functional form of a "production" function g does not have to be specified, nor does g''/g' need to be unbounded. This is important in the view of Assumption 3 and Theorem 3 where for MSNE in L_M we need g'/g'' to be bounded. Hence, our Example 2 can satisfy Assumption 3 as well.

The theorem also provides a *globally stable* successive approximation algorithm which allows us to compute the unique equilibrium, as well as providing the basis for *uniform* error bounds for equilibrium values directly. To see this, from the unique relationship between h^* and p^* (i.e., $h^*(x) = c^*_{p^*}(x)$), we are able to relate theorems on h^* with theorems that concern p^*. To do this, we simply relate iterations on the operators A, with iterations on the operator B as follows:

$$A^{n+1}h_0(x) = c^*_{B^n p_0}(x), \tag{9}$$

with $p_0(x) = \int_I v(h_0(y))\lambda(dy|x)$. Then, by continuity of $p \to c^*_p(x)$ pointwise, we have $A^{n+1}h_0 \to h^*$ pointwise. Since (under Assumption 3) $A^{n+1}h_0 \in L_M$, and L_M is compact, pointwise convergence implies uniform convergence.

These relationships are summarized more formally in the following corollary.

Corollary 1. *Let Assumption 3 and the assumptions of Theorem 4 be satisfied. Then, A has a unique fixed point h^*, and*

$$(\forall h_0 \in L_M{}^\circ) \quad \lim_{n\to\infty} \|h_n - h^*\| = 0, \tag{10}$$

where

$$\|h_{n+1} - h^*\| \leq M_A(1-\tau^{r^n}), \tag{11}$$

where $M_A > 0$ is a positive constant dependent on the choice of h_0.

We finish this section with an example, where condition (4) is not satisfied. Indeed, in this case, we show MSNE are multiple (in particular, three).

Example 3. Let $S=[0,1]$, $u(c) = \sqrt{c}$, and $v(c) = c^4$. Assume that a transition probability $Q(\cdot|x-c,x) = (1-\sqrt{x-c})\delta_0(\cdot) + \sqrt{x-c}\lambda(\cdot|x)$, and $\lambda(\cdot|x) = \mu(\cdot)\chi_G(x) + \chi_{G^c}(x)\nu(\cdot)$, where χ_G is an indicator function. Assume that G is some subset of

[26] Observe that condition (4) is equivalent to: $[\ln(v(c))]' \leq r[\ln(g'(x-c))]' - r[\ln(u'(c))]'$. We thank anonymous referee for suggesting this formulation.

$S := [0,1]$ containing number $\sqrt[4]{0.9}$ and not containing $\sqrt[4]{0.8}$. Assume that ν is supported in G while μ on G^c, and 4th moments of μ and ν respectively, are $M_\mu^4 = 0.8$ and $M_\nu^4 = 0.9$.

We now turn to analyze functional equation (3). Under assumptions of the example we obtain:

$$p(x) = \int_S \left(\frac{y}{1+p^2(y)}\right)^4 \lambda(dy|x) = \int_S \left(\frac{y}{1+p^2(y)}\right)^4 \mu(dy)\chi_G(x) + \int_S \left(\frac{y}{1+p^2(y)}\right)^4 \nu(dy)\chi_{G^c}(x).$$

Observe that solution p must be constant on G and its complement, hence we can put

$$p(x) = \alpha\chi_G(x) + \beta\chi_{G^c}(x). \tag{12}$$

Since μ is concentrated on G^c and ν on G we have

$$p(x) = \int_S \left(\frac{y}{1+\beta^2}\right)^4 \mu(dy)\chi_G(x) + \int_S \left(\frac{y}{1+\alpha^2}\right)^4 \nu(dy)\chi_{G^c}(x), = \frac{M_\mu^4}{(1+\beta^2)^4}\chi_G(x) + \frac{M_\nu^4}{(1+\alpha^2)^4}\chi_{G^c}(x). \tag{13}$$

Combining (12) and (13) we have

$$\begin{cases} \alpha = \dfrac{0.8}{(1+\beta^2)^4}, \\ \beta = \dfrac{0.9}{(1+\alpha^2)^4}. \end{cases}$$

Hence α solves the equation:

$$\alpha\left(1+\left(\frac{0.9}{(1+\alpha^2)^4}\right)^2\right)^4 = 0.8.$$

There are exactly three αs satisfying this equation: $\alpha_1^* \approx 0.547$, $\alpha_2^* \approx 0.705$ and $\alpha_3^* \approx 0.08$. This yields exactly three fixed points $p_1^*(x) = 0.547\chi_G(x) + 0.316\chi_{G^c}(x)$, $p_2^*(x) = 0.705\chi_G(x) + 0.179\chi_G(x)$, and $p_3^*(x) = 0.08\chi_G(x) + 0.877\chi_G(x)$. These fixed points correspond to three different MSNE:

$$c_1^*(x) = \begin{cases} 0.77x & \text{if } x \in G, \\ 0.909x & \text{if } x \notin G, \end{cases}$$

$$c_2^*(x) = \begin{cases} 0.668x & \text{if } x \in G, \\ 0.969x & \text{if } x \notin G, \end{cases}$$

and

$$c_3^*(x) = \begin{cases} 0.993x & \text{if } x \in G, \\ 0.565x & \text{if } x \notin G, \end{cases}$$

4.3. Continuous equilibrium comparative statics

In this section, we consider the question of conditions for continuous comparative statics of an MSNE equilibrium set. Such a condition would be very useful (for example) when estimating deep parameters of the models (e.g., via GMM or some simulated moments method). Such comparative statics are also sufficient to build rigorous applications of calibration methods to the question of sensitivity analysis in our games.

To study this question, we first parameterize primitives of our economy by $\theta \in \Theta$, where Θ is a compact interval in $\mathbb{R}^m$. For each $\theta \in \Theta$, let $u(\cdot,\theta)$, $v(\cdot,\theta)$, $g(\cdot,\theta)$ be functions summarizing preferences and stochastic technologies in the previous sections of the paper (only now, we let them depend on θ), and let the probability measure we use to generate the stochastic transitions on the state be parameterized as $\lambda(\cdot|x,\theta)$. Notice initially that when Assumptions 1–3 are satisfied for any $\theta \in \Theta$, the constant $M = 2 + M_F M_g$ might depend on θ. Denote this dependence on deep parameters by $M(\theta)$, and observe that for compact I, we have $M(\theta) < \infty$ for all $\theta \in \Theta$. Therefore, if we further assume that $\sup_{\theta\in\Theta} M(\theta) \le \overline{M}$ for some constant $\overline{M} > 0$, we let $LNE_{\overline{M}}(\theta)$ denote the set of MSNE belonging to $L_{\overline{M}}$ in the game with parameter θ. For future reference, we often denote this correspondence as a mapping $\theta \to LNE_{\overline{M}}(\theta)$.

We now have the following comparative statics theorem.

Theorem 5. *For each $\theta \in \Theta$, let Assumptions 1–3 be satisfied, and I bounded. Moreover, let the mappings $(c,\theta) \in I^\circ \times \Theta \to u(c,\theta)$, $(c,\theta) \to v(c,\theta)$, $(i,\theta) \in I^\circ \times \Theta \to g(i,\theta)$ be continuous, as well as let the collection $(\theta,x) \to \lambda(\cdot|x,\theta)$ has Feller property. If in addition, we assume $v(\cdot,\cdot)$ is uniformly continuous, and $\sup_{\theta\in\Theta} M(\theta) \le \overline{M}$ for some constant $\overline{M} > 0$, then the correspondence $\theta \to LNE_{\overline{M}}(\theta)$ is upper hemicontinuous (i.e. has a closed graph).*

Corollary 2. *Let assumption of Theorem5 be satisfied. If for all θ, there is a unique MSNE, then the function $\theta \rightarrow LNE_{\overline{M}}(\theta)$ is continuous.*

The theorem (as well as its corollary) is very important in applications. First, obviously, the corollary gives conditions under which continuous sensitivity analysis of the equilibrium set is possible. Such a result is critical in developing conditions under which the simulated moments of the model converge to the actual moments of the model (e.g., Santos and Peralta-Alva, 2005).

Second, given recent work on approximating upper hemicontinuous correspondences (e.g., Beer, 1980; Feng et al., 2009), the theorem implies that one can build step function approximation scheme to approximate the equilibrium correspondence to compute equilibrium comparative statics. For such an algorithm, it will be the case that as the "mesh" of the approximation scheme becomes finer, the approximation scheme will converge pointwise Hausdorff to the mapping $LNE_{\overline{M}}$ (uniform in the case of the corollary to the theorem). Constructing such an approximation scheme is possible because the theorem shows that the equilibrium correspondence $LNE_{\overline{M}}$ is valued in a *compact subset of a function space*, where uniform approximation schemes for arbitrary elements of this function space can easily be constructed (e.g., using various discretization schemes or piecewise linear/constant approximation schemes), as well as the fact that the theorem says the entire set of MSNE moves in an upper hemicontinuous manner. The result is important as it is difficult to imagine how one could obtain such a strong characterization of the set of MSNE using the various alternative methods in the existing literature (i.e., APS or generalized Euler equation methods (henceforth, GEE)).

Third, the theorem provides an exact analog to the correspondence based solutions methods for games (e.g., APS methods) relative to the question of *computable equilibrium comparative statics.* This is in contrast to the existing methods, whose focus is on how to compute the set of Markov or subgame perfect equilibrium that exist at a *particular* parameter, say $\theta \in \Theta$. So, in this sense, our methods provide a new direction for correspondence-based computational methods that are an alternative to the methods in the existing literature.

Finally, returning to the corollary of the theorem, we know of no analog to this theorem in the existing literature using either GEE methods or correspondence based/promised utility methods. In particular, without our geometric approach (which require operators in function spaces to characterize the requisite geometric conditions), even if it is known that the equilibrium correspondence $LNE_{\overline{M}}$ is nonempty valued, it is not known if it is a function. Such a sharp characterization of $LNE_{\overline{M}}$ is needed if one wants to have a great deal of certainty that the comparative static computed actually corresponds to the actual comparative static that arises in the equilibrium of the stochastic game.

4.4. Existence of stationary Markov equilibrium and stochastic equilibrium dynamics

The results stated in Theorems 3 and 4 allow us to further characterize the structure of MSNE for the economies under study. To do this, we first prove a result on the existence for Stationary Markov Equilibrium (SME). We define a Stationary Markov equilibrium to be a pair $h^* \in L_M$ (that is, a pure-strategy MSNE in L_M), and its set of associated invariant distributions on I. We prove the following result:

Theorem 6. *Let Assumptions 1–3 be satisfied, with $g(0)>0$, and I bounded. Assume additionally that $\|g\|<1$, and let $h^* \in L_M$ be a MSNE of the game. Then, the Markov process induced by Q, parameterized by h^* has a unique invariant distribution, and a process started from $x_0 \in I$ converges to this distribution.*

With results of Theorem 4, we have the immediate corollary that is particularly useful in applications:

Corollary 3. *Under Assumptions 1–3 with $g(0)>0$ and I bounded, there exists a SME. If in addition other conditions of Theorem 4 are satisfied, then there exists a unique SME.*

We make two remarks per these results. First, it is important to note that in our model, as $\lambda(\cdot|x)$ is stochastically decreasing, the transition probability is not a special case of that in the work of Amir (1996b). If $\lambda(\cdot|x)$ is stochastically increasing (see e.g. Amir, 1996b) we would easily obtain convergence to an invariant distribution (by the Knaster–Tarski theorem); the problem is its uniqueness would not be guaranteed. Such a situation could be a significant complication in some applications (e.g., in calibration or estimation problems).

Second, apart from previously stated assumptions, in these results, we do require that $g(0)>0$. Many common production functions used in applied work satisfy this condition.[27] We only need this condition for our results on SME, not existence of MSNE. The assumption is required to avoid situations that arise in the literature under existing assumptions (e.g., Nowak, 2006, where one assumes bounded I and $g(0)=0$). In this case, the invariant distribution induced by a MSNE h^* can be unique, but trivial.

[27] For example, many CES production functions. That is, for our economy assuming inelastic labor supply (hence, in equilibrium, $n=1$), one specification for g has $\hat{g}(k,n)=G(g_1(k)+g_2(n))$, where G and g_i are strictly increasing, strictly concave, and smooth, and $g_i(0)=0$. Then $g(0)=\hat{g}(0,1)>0$. For example, if g given by a standard CES production function (used commonly in the real business cycle and macroeconomics literature), we have $\hat{g}(k,n)=[k^\sigma+n^\sigma]^{1/\sigma}$, which has $g(k)=\hat{g}(k,1)$ satisfying our assumption as $g(0)=\hat{g}(0,1)=1>0$. To see why the condition is needed, as we allow $g(0)=0$, given our specification of Q, observe that the unique equilibrium invariant distribution associated with any MSNE will be concentrated at the point $x=0$.

Example 4. Assume 1–3 with I bounded but with $g(0)=0$. Let $\{x_t : t \in N\}$ be a Markov chain generated by $Q(\cdot|x_t - h^*(x_t), x_t)$. We now show that $x_t \to 0$ almost sure and $x_t > 0$ for at most finite number of t.

Let $p_t := Prob(x_t > 0)$. Let $E(\cdot)$ be an expected value operator induced by the chain $\{x_t : t \in N\}$. We show that $\sum_{t=1}^{\infty} p_t < \infty$. Observe that $p_t = 1 - F_t(0)$, where F_t is a distribution function of x_t. Clearly, $F_t(0) := 1 - E(g(x_{t-1} - h(x_{t-1}))) := 1 - \rho_{t-1}$. By our assumptions, we have

$$\begin{aligned}
\rho_t &= E(g(x_t - h(x_t))) \\
&= E\left(\int_I g(x-h(x))\lambda(dx|x_{t-1})g(x_{t-1}-h(x_{t-1}))\right) \\
&\le E\left(\int_I g(x)\lambda(dx|x_{t-1})g(x_{t-1}-h(x_{t-1}))\right) \\
&\le \int_I g(x)\lambda(dx|0)\rho_{t-1}.
\end{aligned} \tag{14}$$

Since $0 < \int_I g(x)\lambda(dx|0) < 1$, we have $\sum_{t=1}^{\infty} \rho_t \le \infty$. Since $p_t = \rho_{t-1}$, by Borel–Cantelli theorem, we obtain the occurrence $\{x_t > 0\}$ holds for the finite number of t almost surely.

The above example shows exactly the content of our productivity assumption $g(0) > 0$. Intuitively, the need for the condition is quite simple: if we let 0 be an absorbing state, given the specification of the noise on compact I, eventually the equilibrium processes will end up in a trivial invariant distribution, delta Dirac concentrated at point 0. Also, the reasoning provided in the above Example 4 can be easily generalized to the case where I is *unbounded*, since by Assumption 2 $\lambda(\cdot|x)$ is stochastically *decreasing*.

Finally, and importantly, note that if we assume that $\lambda(\cdot|x)$ is stochastically increasing and I is unbounded, this reasoning above will *not* work. Therefore, for example, although Amir (1996b) is not characterizing the set of SME of a similar bequest game to ours, we cannot claim that under $g(0)=0$, the set invariant distribution in his class of games would be trivial.

5. Further discussion and extensions

We conclude with a discussion of how our results can be extended. In particular, we discuss how to (i) derive error bounds for an approximation procedure for computing MSNE of the bequest game, (ii) show the uniform convergence of nonstationary equilibria in the finite horizon bequest game to the equilibria in infinite horizon game, (iii) extend our uniqueness result to the case of models with non-separable utility, and finally (iv)present a simple numerical example illustrating all our results derived in the paper.

We begin with the question of uniform approximation schemes for MSNE.

5.1. Computing MSNE

We first construct accurate approximate schemes for MSNE for the economies we consider. In particular, we discuss a simple discretization method for computing fixed points of the operator A (and, hence B via relation (9)) corresponding to a unique MSNE of our bequest economy. Following standard arguments in the literature (e.g., Fox, 1973; Bertsekas, 1975; Hinderer, 2005), and exploiting the Lipschitz and uniform continuity of MSNE, we can calculate uniform error bounds for an approximation of given precision, and we can prove that a discretization procedures converge uniformly to an actual solution as its precision/mesh of the scheme gets arbitrarily large/fine. We consider the case of bounded intervals I.

Consider the following discretization scheme. Partition bounded set I into m mutually disjoint intervals $I_1, I_2, \ldots, I_m$ such that $I = \cup_{i=1}^m I_i$ where $x_i \in I_i$ and $P_m = \{x_1, x_2, \ldots, x_m\}$. Denote by $d_m = \max_{i=1}^m \sup_{x \in I_i} |x - x_i|$, i.e. the maximal grid size. Consider a function h_m, as well as an operator A_m, where h_m is a piecewise-constant approximation (i.e. a step function approximation) of h defined by[28]

$$(\forall x \in P_m) \quad h_m(x) = h(x),$$
$$(\forall x \in I_i) \quad h_m(x) = h(x_i),$$

and, similarly, $A_m h_m$ is a piecewise-constant approximation to Ah_m defined by

$$(\forall x \in P_m) \quad A_m h_m(x) = Ah_m(x),$$
$$(\forall x \in I_i) \quad A_m h_m(x) = A_m h_m(x_i).$$

So, the approximation is the following: the approximate function is set equal to the original function on the grid of the approximation, and we extend the approximation's definition to the whole compact interval I by defining the approximation to be constant in each of the subintervals I_i. This is a standard piecewise constant discretization scheme.

[28] We choose a piecewise constant approximation scheme because it is arguably the simplest approximation scheme on can imagine. Obviously, similar results are available for piecewise linear, splines, and some polynomial schemes.

Now, for any $h^0 \in L_M$, let the approximation of h^0 be given by h_m^0, and define A_m^n be an n-th iteration of the approximate operator A_m from h_m^0.

Theorem 7. *Let Assumption3 and those of Theorem 4 be satisfied with $M_F < \infty$, and $d_m \to 0$ as $m \to \infty$. Then, for any $h^0 \in L_M{}^\circ$:*

$$\lim_{n\to\infty}\lim_{m\to\infty} \|A_m^n h_m^0 - h^*\| = 0. \tag{15}$$

where $h^* \in L_M{}^\circ$ is the unique fixed point of operator A. Moreover, we have the following estimate for an approximation error:

$$(\forall n,m \in \mathbb{N}) \quad \|A_m^n h_m^0 - h^*\| \le (n+1)d_m M + M_A(1-\tau^{r^n}), \tag{16}$$

where $1 > r > 0$ and $M_A > 0, 1 > \tau > 0$ are constants that are dependent on a choice of $h^0 \in L_M{}^\circ$.

Observe that the inequalities in (10) and (11) imply that operator A has properties similar to a contraction mapping. Indeed, appealing to versions of the converse to Banach's contraction mapping theorem (namely, Janos, 1967 or more recently Hitzler and Seda, 2001), one can show that there exists a metric that induces an *equivalent* topology as the sup norm and under which the operator A is a *contraction*. Further, by the main theorem in Leader (1982), there exist direct links between iterations of the operator A in the original metric (i.e., sup metric) and the induced iterations of A under the equivalent metric under which it is now a contraction. This link proves a critical step when calculating error bounds of our approximation.

5.2. Finite horizon stochastic games

We now can show how MSNE is the limit of (nonstationary) Markov Nash Equilibrium in finite horizon versions of our stochastic game. In particular, we provide conditions when the unique MSNE is the uniform limit of the collection of nonstationary Markov Nash equilibrium for finite horizon versions of our game. Such conditions are not known in the existing literature.

To do this, first consider a finite horizon case of our bequest game (i.e. an economy populated by T generations, each with preferences $(\forall t < T)\ u(c_t) + g(x-c_t) \int v(c_{t+1}(y))\lambda(dy|x)$, and the terminal generation with payoff $u(c_T)$). The results on existence and uniqueness of nonstationary equilibria in this class of games are well known (e.g. see Amir, 1996a). Now, consider the limiting behavior of Markovian Nash equilibrium in these games. More succinctly, if c_T^* is the optimal strategy of a first generation in the T-horizon bequest game in a Markovian Nash equilibrium, can we give conditions that guarantee (i) uniqueness of MSNE in the infinite horizon game, say h^*, such that (ii) we have $\lim_{T\to\infty} c_T^* = h^*$ uniformly. We give those conditions in the following lemma:

Lemma 8. *Assume 1–3 as well as conditions in Theorem 4. Then h^* is the unique MSNE of the infinite horizon bequest game, and we have the nonstationary pure strategy Nash equilibrium in the finite horizon game satisfying*

$$\|c_T^* - h^*\| \to 0.$$

5.3. Extensions to economies with non-separable utility via mixed-monotone operators

We next extend our uniqueness results for MSNE to economies where each generation has *non-separable* utility. For this, we appeal to a branch of fixed point theory that has not found any application in economics (namely, the fixed point theory for *mixed monotone* operators).

We begin by defining a mixed monotone operator in the context of our application. Consider an operator $f: X \times X \to X$, where X is a partially ordered set. We say $f(x,y)$ is *mixed-monotone* if (i) for each $y \in X$, $x \to f(x,y)$ is increasing and (ii) for each $x \in X$, $y \to f(x,y)$ is decreasing. We shall say x^* is a fixed point for our mixed-monotone operator if $f(x^*,x^*) = x^*$. For the existence and computation of fixed points of f, we will exploit important results on existence and uniqueness of fixed points for mixed monotone operators at each stage that are due to Guo et al. (2004). These fixed point results exploit geometric conditions that imply that "two stage" iterations from least and greatest elements of X will converge to unique fixed points. We shall show these geometric conditions can be checked in important economic applications.

To develop our arguments for this section, we shall maintain the conditions on technology in Assumption 2, but replace Assumption 1 with the following new condition:

Assumption 4 (*Preferences*). The utility function satisfy:

- $U(c)$ is given by $U(c_1,c_2) = u(c_1) + v(c_1)\nu(c_2)$ where $u,v,\nu : \mathbb{R}_+ \to \mathbb{R}_+$, are strictly increasing and continuous. Moreover u,v are continuously differentiable and are strictly concave,
- $u(0)=0$, $v(0)=0$, $\nu(0)=0$,
- $\lim_{a\to 0} u'(a) = \infty$ and $(\forall a \in I, a > 0) 0 < u'(a) < \infty$,
- $\lim_{a\to 0} g'(a) = \infty$ and $(\forall a \in I, a > 0) 0 < g'(a) < \infty$.

Using Lemma 1 from Nowak (2006), for a continuation strategy for the successor generation $h \in H$, a standard argument shows that objective function for the current generation:

$$\mathcal{W}(c,x,h) = u(c) + v(c)g(x-c)\int_I \nu(h(y))\lambda(dy|x)$$

is well-defined, strictly concave in c on $I(x)$. Therefore, at each $h \in H$, there is a unique best response for the current generation. Also, note that as u,v and g are each continuously differentiable, we can again use the linearization $(\partial\mathcal{W}/\partial c)(c,x,h)$ to define a mapping $\zeta(c,x,h)$ as follows:

$$\zeta(c,x,h) := u'(c) + v'(c)g(x-c)\int_I \nu(h(y))\lambda(dy|x) - v(c)g'(x-c)\int_I \nu(h(y))\lambda(dy|x).$$

Using ζ as before, define a new mapping $\varsigma(c,x,h_1,h_2)$:

$$\varsigma(c,x,h_1,h_2) = u'(c) + v'(c)g(x-c)\int_I \nu(h_1(y))\lambda(dy|x) - v(c)g'(x-c)\int_I \nu(h_2(y))\lambda(dy|x),$$

where $h_1,h_2 \in H$. Notice, we have decomposed the continuation structure of the game into a pair of functions (h_1,h_2). We can use this decomposition in ς to define a *mixed-monotone* operator.

To see this, first observe that ς is strictly decreasing with c and h_2 and strictly increasing with h_1. Define an operator C on $\mathcal{C}(I) \times \mathcal{C}(I)$ for $x>0$ by $C(h_1,\underline{\theta}) = \overline{\theta}$ (where $\underline{\theta}(y) \equiv 0$ and $\overline{\theta}(y) = y$), and if $h_2 > \underline{\theta}$ then $C(h_1,h_2)$ is an argument c which solves the equation $\varsigma(c,x,h_1,h_2) = 0$. Observe, as ς is strictly decreasing with c, the Inada conditions guarantee that C is well defined. Moreover, as ς is continuous in x and c, we have $C : \mathcal{C}(I) \times \mathcal{C}(I) \to \mathcal{C}(I)$. Finally, Inada conditions imply that $C(h_1,h_2) \neq \underline{\theta}$. The fact that C is mixed monotone is straightforward. Also, note that as opposed to the method used in Section 4.2, here we define an operator not on values but on the first order condition.

We now extend our main result of the previous section to our new setting. In this case, instead of proving the general theorem on existence using implied elasticities of current vs. continuation payoffs, for simplicity, we use power utility functions (similar to Example 2) to show conditions for the uniqueness of MSNE with nonseparable payoffs. Recall, power utility functions are often found in applied work.

Proposition 1. *Let $u(c_1) = c_1^\gamma$, $v(c_1) = c_1^\alpha$, and $\nu(c_2) = c_2^\beta$, where $0<\alpha<1$, $0<\beta<1$, $0<\gamma<1$ with I bounded. If Assumptions 2 and 4 on bounded I are satisfied with $\alpha+2\beta<1$, $\gamma \le \alpha$, then, C has a unique fixed point on $\mathcal{C}(I)$ and condition (A.5) holds.*

5.4. Example

We conclude the paper by presenting a simple numerical example that illustrates all results of our paper. In particular, we consider the special case of power utility and Cobb–Douglas production. This class of primitives has found extensive use in applied work in macroeconomics, for example.

Example 5. Consider the following environment: let the state space be $I=[0,1]$, preferences given by a time separable utility $U(c_1,c_2) = c_1^\alpha + c_2^\beta$, the stochastic transition on output is given by $Q(\cdot|x-c,x) = (1-(x-c)^\gamma)\delta_0(\cdot) + (x-c)^\gamma\lambda(\cdot|x)$, where δ_0 is a delta Dirac concentrated at $x=0$, and the "production" function $(x-c)^\gamma$ is of a standard Cobb–Douglas form, with the measure λ having a cdf given by $(1-e^{-(2-x)y})/(1-e^{-(2-x)})$. Under these assumptions, 0 is an absorbing state of Q. Let $1>\alpha>0, 1>\beta>0, 1>\gamma>0$.

For this economy, our main existence theorems (Theorems 3 and 4) show the set of MSNE is nonempty in a class of Lipschitzian functions and contains no ordered elements in the standard pointwise partial order. Further, if $\alpha+\beta<1$, then there exist a unique MSNE given by a Lipschitz continuous function, and is unique relative to a class of bounded measurable functions. Moreover, this consumption policy is just the pointwise limit of the sequence of consumption policies for finite horizon version of our economy (Lemma 4). By Corollary 2, we have continuous equilibrium comparative statics (i.e., this MSNE is continuous in the deep parameters). Finally, we can easily approximate MSNE consumption policy by a piecewise-constant approximation scheme to any arbitrary level of accuracy (in the sup norm) by using a simple Picard iteration scheme (Theorem 7). Proposition 1, then, extends these results to the case of non-separable utilities.

The numerical results of our calculations for this example[29] are presented in Fig. 1. In the left panel, we assume the capital share $\gamma = 0.33$, and vary the preference parameters α and β. In the right panel, we let the preference parameters be $\alpha = 0.6$ and $\beta = 0.3$, and vary γ. Sensitivity analysis shows the large discrepancies in consumption values and more importantly consumption function slopes.

It bears mentioning that correspondence-based methods (e.g., APS methods) do not apply to this example (e.g., we do not assume discounting). Further, GEE methods are not needed (as necessary and sufficient Euler equations can be used to characterize best replies), nor are their application necessarily justified, as MSNE are not necessarily smooth.

[29] MATLAB program implementing our numerical procedure is available from authors upon request.

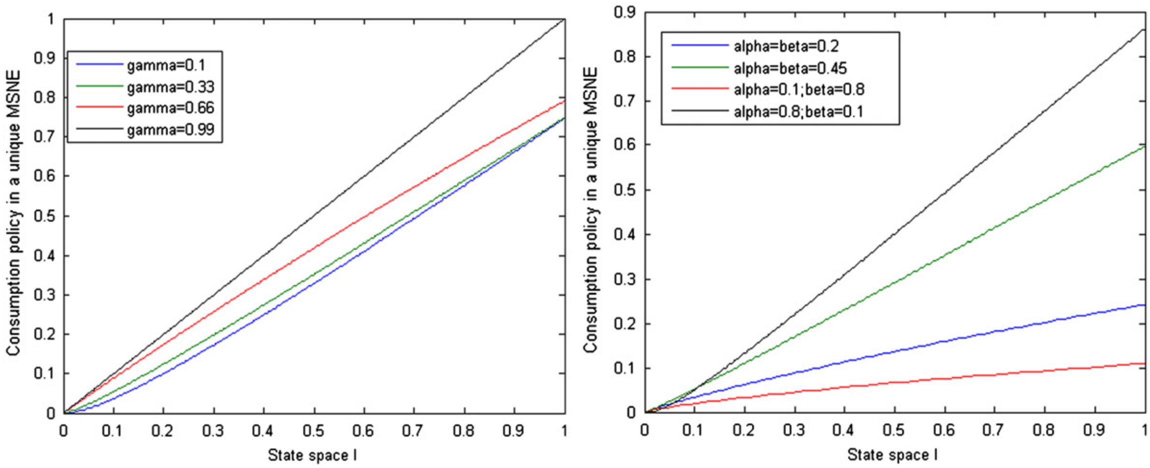

Fig. 1. Consumption policy in a unique MSNE. Left panel: parameter $\gamma = 0.33$ and various α,β. Right panel: parameters $\alpha = 0.6, \beta = 0.3$ and various γ.

6. Conclusion

This paper proposes an order-theoretic method for proving the existence of Markov stationary Nash equilibria (MSNE) in games of intergenerational altruism, as well as new methods for constructing nontrivial SME. Under additional conditions, using new geometrical methods in operator theory, we are able to give simple conditions for the existence of a unique MSNE, as well as provide a catalog of globally stable iterative procedures for computing nontrivial SME. Our theorems are sharp, in the sense that we can provide explicit examples when uniqueness conditions are met, and when they fail. As the methods are also constructive, we are able to provide a detailed accounting of the relationship between approximate and actual MSNE for the economies we study. Our methods also work for models with non-separable lifecycle utility. For this case, we use a "mixed-monotone" method, which has not been applied in the existing literature on stochastic games.

The main results of the paper (e.g., Theorems 3 and 4) present sufficient conditions for existence and uniqueness of MSNE that are often met in applied work in intergenerational models of dynamic equilibrium (e.g., overlapping generations models in macroeconomics). We argue that in applications, global stability is a key property of the set of MSNE, and we are able to prove results on continuity of the set of MSNE in deep parameters (a result very useful for the simulation and estimation of the models at hand). Finally, using fixed points methods for mappings in abstract cones, we are able to construct a sequence of policies converging uniformly to the unique MSNE of the economy. The results can be extended to asymptotic results for approximate solutions via simple discretization arguments, and we can then obtain uniform error bounds for computing unique MSNE. We stress that the methods and examples presented in the paper can be generalized and used to study the broad class of overlapping generations macroeconomies with partial commitment.

Many key issues, though, still remain to be studied in future work. For example, one can ask if it is possible to design a decentralized OLG economy without commitment that corresponds to decentralization of the equilibria in bequest game under study. If this is possible, the results here could provide a tractable dynamic general equilibrium model without commitment that would be useful for applied work that study lifecycle models with limited commitment. In principle, one could compute equilibrium prices, and prove the existence of a unique decentralized stationary Markov equilibrium. Additionally, one could add labor/leisure choice into the environment (see Balbus et al., 2012c). We feel that these two steps are necessary in order to bring the models with partial commitment that are common in macroeconomic applications to the level of analysis which is now common with models lacking such strategic interactions across generations.

Finally, as mentioned in the introduction, we believe that our monotone value function operator methods can be applied to other classes of economies, including time-consistency games such a policy games and models with hyperbolic discounting. For example, in Balbus et al. (2012b), we show how our monotone operator methods can be used to compute optimal, time-consistent consumption policy of a quasi-hyperbolic consumers under uncertainty facing borrowing constraints. Additionally, in Balbus et al. (2012a), we prove a number of theorems allowing for a constructive study of equilibria of stochastic supermodular games including such models as dynamic Bertrand models, dynamic R&D models, dynamic public good games or time-consistent public policies.

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Appendix A. Definitions and abstract fixed point theorems

Here we provide few definitions, as well as state a useful theorem that we use in our discussion on unique MSNE.[30]

Definition 1. Let E be a real Banach space and $P \subseteq E$ be a nonempty, closed, convex set. Then

- P is called a cone if it satisfies three conditions: (i) $p \in P, \epsilon > 0 \Rightarrow \epsilon p \in P$; (ii) $p \in P, -p \in P \Rightarrow p = \theta$, where θ is a zero element of; and (iii) $p \in P$, $q \in P \Rightarrow p+q \in P$,
- suppose P is a cone in E and $P^\circ \neq \emptyset$, where P° denotes the set of interior points of P, we say that P is a solid cone,
- a cone P is regular iff each increasing sequence which has an upper bound in order has a limit,
- a cone P is said to be normal if there exists a constant $N > 0$ such that
$$(\forall p_1, p_2 \in P) \quad \underline{\theta} \le p_1 \le p_2 \Rightarrow \|p_1\| \le N\|p_2\|.$$

Theorem 9 (Guo et al., 2004). *Let P be a normal solid cone in a real Banach space with partial ordering $\le$ and $B : P^\circ \to P^\circ$ be a decreasing operator (i.e. if $p_1 < p_2 \in P$ then $Bp_2 \le Bp_1$) satisfying*

$$(\exists r, 0 < r < 1)(\forall p \in P^\circ)(\forall t, 0 < t < 1) \quad t^r B(tp) \le Bp, \tag{A.1}$$

then B has a unique fixed point p^* in P° and

$$(\forall p_0 \in P^\circ) \quad \lim_{n\to\infty} \|p_n - p^*\| \to 0, \tag{A.2}$$

where $(\forall n \ge 1) p_n = B(p_{n-1})$. Moreover we have the following estimate of convergence rate:

$$\|p_n - p^*\| \le M_B(1-\tau^{r^n}), \tag{A.3}$$

where $M_B > 0$ and $0 < \tau < 1$ are positive constants dependent on the choice of p_0.

It is important to stress that Guo et al. (2004) establish uniqueness results under weaker conditions than we use in our work (A.1). Rather, we use a stronger version of their result which also guarantees the other conclusions of Theorem 9.

Definition 2. Let P be a cone in real Banach space with partial ordering $\le$ and operator $C : P \times P \to P$. If C is increasing with the first argument and decreasing with the second, i.e. $(\forall p', p, q', q \in P)(p' \ge p)(q' \ge q)$, we have $C(p', q) \ge C(p, q)$ and $C(p, q) \ge C(p, q')$, we say that C is a mixed monotone operator. If $\exists p^* \in P$ such that $C(p^*, p^*) = p^*$, then p^* is called a fixed point of C.

Theorem 10. *Let P be a normal solid cone in real Banach space with partial ordering $\le$ and $C : P^\circ \times P^\circ \to P^\circ$ be a mixed monotone operator. Assume that there exists a constant $0 < r < 1$ such that*

$$(\forall 0 < t < 1)(\forall p, q \in P^\circ) \quad C(tp, t^{-1}q) \ge t^r C(p, q). \tag{A.4}$$

Then, C has a unique fixed point $p^* \in P^\circ$, and for any $p_0, q_0 \in P^\circ$, we have

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = p^*, \tag{A.5}$$

where $\forall n \ge 1$, $p_n = C(p_{n-1}, q_{n-1})$ and $q_n = C(q_{n-1}, p_{n-1})$.

Appendix B. Proofs

Proof of Lemma 1. First we show that function $x \to p_f(x)$ is differentiable for any continuous bounded and increasing function f. Without loss of generality, assume $\lim_{y\to\infty} f(y) = 1$. Observe that p_f is a Lebesgue–Stieltjes integral on a form:

$$p_f(x) = \int_I f(y) F_\lambda(dy|x).$$

[30] See Theorem 3.2.5 in Chapter 3.2 of Guo et al. (2004).

Since f is continuous increasing and bounded from above, and $F_\lambda(\cdot|y)$ is right-continuous, by Theorem 21.67 in Hewitt and Stromberg (1965), we can integrate it by parts to arrive at

$$p_f(x) = 1 - \int_I F_\lambda(y|x) df(y). \tag{B.1}$$

Calculating the derivative of the left side of (B.1), and letting $h_n \to 0$ be a sequence, observe that as $F_\lambda(y|x)$ is differentiable in the interval containing x, there exists a value $\theta_y^n \in (0,1)$ such that

$$\left| \frac{F_\lambda(y|x+h_n) - F_\lambda(y|x)}{h_n} \right| = -\frac{\partial}{\partial x} F_\lambda(y|x+\theta_y^n h_n)$$
$$\le M_F^W (1 - F_\lambda(y|x+\theta_y^n h_n)) \le M_F^W.$$

where the last inequality follows from Assumption 3. Since $\int_I M_F^W \, df(y) < \infty$, by dominated convergence theorem:

$$\lim_{n\to\infty} \int_I \frac{F_\lambda(y|x+h_n) - F_\lambda(y|x)}{h_n} \, df(y) = \int_I \frac{\partial}{\partial x} F(y|x) \, df(y).$$

As a result, by (B.1), we have

$$p_f'(x) = -\int_I \frac{\partial}{\partial x} F_\lambda(y|x) \, df(y). \tag{B.2}$$

Combining with (B.2), Assumption 3 and (B.1), we have $|p_f'(x)| \le M_F^W p_f(x)$ □

Proof of Lemma 2. *Step*1: We prove (i). Without loss of generality assume $S = \infty$. We first construct $H_0 \subset H$ such that H_0 contains increasing functions and is compact in the $\stackrel{u}{\to}$ topology. This set will take a form $H_0 := \mathcal{L}_\mathcal{M}$ for some positive valued function $\mathcal{M}$. Before finding this function, we show that such a space $\mathcal{L}_\mathcal{M}$ is compact with endowed topology $\stackrel{u}{\to}$. Let $h \in \mathcal{L}_\mathcal{M}$ be arbitrary. Note that for all $y \in I$ $0 \le h(y) \le y$. Next observe that $\mathcal{L}_\mathcal{M}$ is equicontinuous for all selections of $\mathcal{M}$. Take $y_0 \in I^0$ and sufficiently small $\epsilon \in (0, y_0/2)$. Define $W_0 := [y_0/2, (3/2)y_0]$. Clearly y_0 is interior point of W_0. Let $M_0 := \mathcal{M}(W_0)$. Observe $[y_0-\epsilon, y_0+\epsilon] \subset W_0$. Then, if we take $\delta < \min(\epsilon/M_0, \epsilon)$, then $\sup_{h\in\mathcal{L}_\mathcal{M}} |h(y_0) - h(y')| < \epsilon$ for all $y' \in (y_0-\delta, y_0+\delta)$. For $y_0 = 0$ we have $\sup_{h\in\mathcal{L}_\mathcal{M}} |h(y') - h(0)| = \sup_{h\in\mathcal{L}_\mathcal{M}} h(y') \le y' < \epsilon$ for $\delta = \epsilon$. Hence $\mathcal{L}_\mathcal{M}$ is equicontinuous. Clearly $\mathcal{L}_\mathcal{M}$ is pointwise closed, hence by the Arzela–Ascoli theorem (e.g., Kelley, 1955, Theorem 17, p. 233), $\mathcal{L}_\mathcal{M}$ is compact in the topology $\stackrel{u}{\to}$.

Let $x \in I, x > 0$, and $h \in H$ be increasing. Clearly Ah is well-defined (i.e., nonempty and single-valued). We now find a function $\mathcal{M}$ such that $Ah \in \mathcal{L}_\mathcal{M}$. First, we show that $Ah(\cdot)$ is increasing whenever h is. As (by Assumptions 2 and 3) the function $x \to \zeta(c,x,h)$ (ζ is defined in (2)) is increasing and $c \to \zeta(c,x,h)$ (by Assumptions 1 and 2) is strictly decreasing, there exists x_1 and x_2 such that $0 \le x_1 < x_2 \le S$, and such that if $x \in (0, x_1]$ then $Ah(x) = 0$, if $x \in [x_2, S)$ then $Ah(x) = x$, and if $x \in (x_1, x_2)$ then $Ah(x)$ is a zero of the function $\zeta(\cdot, x, h)$ (i.e. the argument for which this function reaches 0). If $(0, x_1] = \emptyset$, then $x_1 = 0$ and if $(x_2, S) = \emptyset$, then $x_2 = S$. Moreover, $Ah(\cdot)$ is increasing and continuous on intervals $(0, x_1]$, (x_1, x_2) and $[x_2, S)$. As $Ah(x_1) = 0$ and $Ah(x_2) = x_2$, hence for each $x \in [x_1, x_2]$ we have $Ah(x_1) \le Ah(x) \le Ah(x_2)$, which implies that $Ah(\cdot)$ is increasing on all I. If $(0, x_1] = \emptyset$ or $(x_2, S) = \emptyset$, then this monotonicity holds as well.

To see $Ah(\cdot)$ is Lipschitz on all $W \in \mathcal{C}_0$ define $w(c,x,p) := u(c) + p(x)g(x-c)$ for some given continuous, decreasing, nonnegative function p. Recall $\mathcal{W}(c,x,h) = w(c,x,p_{v\circ h})$, where p_f was defined in Section 4.1. Denote the unique argument maximizing w with respect to c to be $c^*(x)$. Also let c_0 denote a zero element of a function $(\partial/\partial c)w(c,x,p)$, if it exists, and observe that

$$c^*(x) = \begin{cases} x & \text{if } u'(x) - p(x)g'(0) > 0, \\ c_0(x) & \text{if } u'(0) - p(x)g'(x) > 0 > u'(x) - p(x)g'(0), \\ 0 & \text{if } u'(0) - p(x)g'(x) < 0, \end{cases}$$

following from concavity of u and g.[31] Since $0 \le c_0(x) \le x$ there exist points $x_i \in I$, $i = 1,2$ such that $0 \le x_1 < x_2 < S$ such that $c^*(x) = x$ for $x \in (x_2, S)$, $c^*(x) = c_0(x)$ for $x \in [x_1, x_2]$ and $c^*(x) = 0$ for $x \in [0, x_1)$. Note that $I_2 := (x_2, S)$ or $I_1 := [0, x_1)$ can be empty sets. Without loss of generality assume that both sets are nonempty. Note that on I_1 c^* is Lipschitz continuous with a constant 0 and on I_2 c^* has a Lipschitz constant 1. It is sufficient to show that c_0 is also Lipschitz continuous on all compact subsets $W \subset (x_1, x_2)$. Note that on (x_1, x_2) $c_0(x) \in (0, x)$ and $p(x) > 0$. Hence by Implicit Function Theorem, the derivative c_0' exists and

$$c_0'(x) = \frac{p(x)g''(x-c_0(x)) + p'(x)g'(x-c_0(x))}{u''(c_0(x)) + p(x)g''(x-c_0(x))}.$$

We now put $p(x) = \int_I v(h(y))\lambda(dy|x)$ for some increasing $h \in H\backslash\{0\}$ such that (x_1, x_2) is a nonempty set. Let $x \in W$. By Lemma 1 we have $|p'(x)/p(x)| \le M_F^W$. Further, we also have

$$|c_0'(x)| = \frac{p(x)g''(x-c_0(x))}{u''(c_0(x)) + p(x)g''(x-c_0(x))} + \left| \frac{p'(x)g'(x-c_0(x))}{u''(c_0(x)) + p(x)g''(x-c_0(x))} \right|$$

[31] We write $u'(0) := \lim_{c\to 0^+} u'(c)$ for short.

$$\le 1+\frac{-p'(x)g'(x-c_0(x))}{-u''(c_0(x))-p(x)g''(x-c_0(x))} \le 1+\frac{-p'(x)g'(x-c_0(x))}{-p(x)g''(x-c_0(x))}$$
$$\le 1+M_F^W M_g<\infty,$$

where M_g is a bound of $|g'/g''|$, and these constants exist by Assumption 3. Hence, $Ah(\cdot)$ is Lipschitz continuous on all compact subsets $W\subset(s_1,s_2)$ with modulus $1+M_F^W M_g$. To show it is Lipschitz continuous on all compact W it is sufficient to show it is continuous. But it is easy to verify that $x\to\int_I v(h(y))\lambda(dy|x)$ is continuous, as h is continuous. Hence, the continuity of $Ah(\cdot)$ follows from continuity of u and g and Berge's maximum theorem. Therefore, $c^*(x)$ is a Lipschitz continuous function with Lipschitz constant $2+M_F^W M_g$ i.e. sum of the Lipschitz constant on subintervals. Setting $\mathcal{M}(W):=2+M_F^W M_g$ we conclude that $Ah\in\mathcal{L}_{\mathcal{M}}$. Obviously, if we take a function $h\in L_M$ and $p(x)$ such that $(x_1,x_2)=\emptyset$, then $Ah=\underline{\theta}$ or $Ah=\overline{\theta}$ and $A(h)\in\mathcal{L}_{\mathcal{M}}$ as well. This implies that A maps $\mathcal{L}_{\mathcal{M}}$ into itself.

We next show that A is continuous (in $\stackrel{u}{\to}$ convergence) on $\mathcal{L}_{\mathcal{M}}$. We have $Ah_n\to Ah$ pointwise when $h_n\to h$, by continuity of functions u,g,v and Lebesgue's dominated convergence theorem. Since $\mathcal{L}_{\mathcal{M}}$ is equicontinuous, the topology of pointwise and $\stackrel{u}{\to}$ convergence coincide in $\mathcal{L}_{\mathcal{M}}$ hence, $Ah_n\stackrel{u}{\to}Ah$ uniformly on all compact subsets of I.

Finally, A is decreasing. Let $h_1\le h_2$. Since $h\to\zeta(c,x,h)$ is decreasing we have $\zeta(c,x,h_1)\ge\zeta(c,x,h_2)$. If $Ah_2(x)=0$ the hypothesis holds trivially. Let $Ah_2(x)\in(0,x]$. Then $\zeta(Ah_2(x),x,h_2)\ge 0$. If $\zeta(Ah_2(x),x,h_2)=0$, and $\zeta(Ah_1(x),x,h_1)=0$. then $\zeta(Ah_2(x),x,h_1)\ge 0$. In this case, $\zeta(\cdot,x,h_1)$ is decreasing, so we obtain $Ah_1\ge Ah_2$. If $\zeta(Ah_1(x),x,h_1)>0$, we immediately obtain $Ah_1(x)=x$, and $Ah_1\ge Ah_2$ as well. Finally, if $Ah_2(x)=x$, then for all $c\in(0,x)$, we have $0\le\zeta(c,x,h_2)\le\zeta(c,x,h_1)$ and $Ah_1(x)=x$ as well.

*Step*2: We prove (ii). We repeat the reasoning from the previous step. We just need to show that there exists a compact set say $H_0\subset H$ such that $A(H_0)\subset H_0$. This is $H_0:=L_M$ for $M:=2+M_gM_F$. Repeating reasoning from the previous step we obtain $|c_0'(x)|\le 1+M_FM_g$ for $x\in(x_1,x_2)$. □

Proof of Theorem 3. *Step*1: Proof of (i). Let H_0 be a set from Lemma 2. As it was mentioned before H_0 is compact in $\stackrel{u}{\to}$. Clearly H_0 is convex. Lemma 2 asserts that A is continuous. Hence, by Schauder's theorem (e.g., Kuratowski, 1966, p. 544), $\Psi_A\subset H_0$ is nonempty. Suppose now that Ψ_A has two ordered fixed points in L_M say, $h,\hat{h}\in\Psi_A$. Then for all $x\in I$, $h(x)\le\hat{h}(x)$. By monotonicity property of the operator A (Lemma 2) we obtain $h(x)=Ah(x)\ge A\hat{h}(x)=\hat{h}(x)$. Hence $h(x)=\hat{h}(x)$. Hence each pair of the ordered fixed points is a pair of identical elements. Therefore, Ψ_A is antichained.

*Step*2: Proof of (ii). Let M be a number from Lemma 2. By the proof of Lemma 2 set L_M is closed, relatively compact and hence compact (in topology $\stackrel{u}{\to}$). The convexity of L_M is obvious. Lemma 2 asserts that A is continuous. Hence, by Schauder's theorem (e.g., Kuratowski, 1966, p. 544), $\Psi_A\subset L_M$ is nonempty. We repeat reasoning from previous step to obtain that Ψ_A is antichain. □

Proof of Corollary 1. Fix $x\in I$ and for any nonnegative constant p consider $l(p)=c_p^*(x)$ i.e. consider l as a function from $\mathbb{R}$ to $\mathbb{R}$. Let us take $p\in[0,\overline{p}]$ with $\overline{p}=\int_I v(y)\lambda(dy|0)$. We show that l is a Lipschitz continuous function on $[0,\overline{p}]$. Since $p\to c_p^*(x)$ is decreasing and continuous, hence there exist number η such that $c_p^*(x)=x$ for all $p\in[0,\eta]$, and $c_p^*(x)\in(0,x)$ for all $p\in(\eta,\overline{p}]$. Inada condition on u guarantees that $c_p^*(x)>0$, whenever $p>0$. Without loss of generality assume that $0<\eta<S$. With fixed x we have $l'(p):=(\partial/\partial p)c_p^*(x)=0$ for $p\in[0,\eta]$. Let $p\in(\eta,S)$. Then $(\partial/\partial p)\omega(l(p),x,p)=0$. By Implicit Function Theorem we have

$$|l'(p)|=\frac{-g'(x-l(p))}{u''(l(p))+pg''(x-l(p))}\le\frac{-g'(x-l(p))}{pg''(x-l(p))}\frac{pg''(x-l(p))}{u''(l(p))+pg''(x-l(p))}\le\frac{M_g}{p}.$$

Hence, for all $\epsilon>0$, and $p_1,p_2\in[\epsilon,\overline{p}]$, we have

$$|l(p_1)-l(p_2)|\le\frac{M_g}{\epsilon}|p_1-p_2|. \tag{B.3}$$

Let $p_{\max}(x):=\int_I v(y)\lambda(dy|x)$. Clearly it is a decreasing and continuous function. Let

$$p_{\max}^1(x):=B(p_{\max})(x)=\int_I v(c_{p_{\max}}^*(y))\lambda(dy|x).$$

Since $\lim_{c\to 0^+}u'(c)=\infty$, $c_{p_{\max}}^*(y)>0$ for all $y>0$. Noting that $p_{\max}^1(\cdot)$ is continuous on compact I (as $p_{\max}$, is continuous), we conclude that $\min_{x\in I}p_{\max}^1(x)>0$. Therefore

$$p_{\max}^1(x)\ge\overline{\epsilon}:=\min_{x\in I}p_{\max}^1(x)>0.$$

Consider a sequence p_n defined in (6). Note that $B(p_0)=p_1\le p_{\max}=B(\underline{\theta})$ as by Theorem 4 operator B is decreasing. Suppose $p_{\max}\ge p_n\ge p_{\max}^1$ for some n. Applying B we obtain $p_{\max}^1\le p_{n+1}$. This implies that $p^n(x)\ge p_{\max}^1(x)\ge\overline{\epsilon}$ for all n and x. By definition of l we have $l(B^np_0)=c_{B^np_0}^*(x)=A^{n+1}(h_0)(x)$. Therefore,

$$|A^{n+1}(h_0)(x)-h^*(x)|=|c_{B^np_0}^*(x)-c_{p^*}^*(x)|\le\frac{M_g}{\overline{\epsilon}}|B^np_0(x)-p^*(x)|\le M_A|1-\tau^{r^n}|,$$

with $M_A:=(M_g/\overline{\epsilon})M_B$. The last inequality follows from definition of $l(\cdot)$ and (B.3). □

Proof of Theorem 5. Let $h(\cdot|\theta_n) \in LNE_{\overline{M}}(\theta_n)$, $\theta_n \to \theta$ and $h(\cdot|\theta_n) \xrightarrow{u} h_0$. We show that $h_0 \in LNE_{\overline{M}}(\theta)$. From definition of $h(\cdot|\theta_n)$ we have

$$u(h(x|\theta_n)) + \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n)g(x-h(x|\theta_n)) \tag{B.4}$$

$$\geq u(c) + \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n)g(x-c), \tag{B.5}$$

for all $c \in [0,x]$. For fixed x we need to show that the convergence

$$J_n := \int_I v(h(y|\theta_n),\theta_n)\lambda(dy|x,\theta_n) \to \int_I v(h_0(y),\theta)\lambda(dy|x,\theta) := J \tag{B.6}$$

is satisfied. We have

$$|J_n - J| \leq \int_I |v(h(y|\theta_n),\theta_n) - v(h_0(y),\theta)|\lambda(dy|x,\theta_n) \tag{B.7}$$

$$+\left|\int_I v(h_0(y),\theta)\lambda(dy|x,\theta_n) - \int_I v(h_0(y),\theta)\lambda(dy|x,\theta)\right|. \tag{B.8}$$

Let $\delta_n := \int_I v(h_0(y),\theta)\lambda(dy|x,\theta_n) - \int_I v(h_0(y),\theta)\lambda(dy|x,\theta)$. Since $v(h_0(y))$ is continuous; hence, by Feller property of $\lambda(\cdot|x,\theta)$, we obtain $\delta_n \to 0$. As $v(\cdot,\cdot)$ is uniformly continuous and $h(\cdot|\theta_n) \xrightarrow{u} h_0(\cdot)$, hence $v(h(\cdot|\theta_n),\theta_n) \xrightarrow{u} v(h_0(\cdot),\theta)$. Since I is bounded, for all $\varepsilon > 0$ there is n_ε such that for all $n > n_\varepsilon$, we have

$$\|v(h(\cdot|\theta_n),\theta_n) - v(h_0(\cdot),\theta)\| < \varepsilon.$$

Therefore, for $n > n_\varepsilon$ we have

$$|J_n - J| \leq \varepsilon + \delta_n.$$

Taking a limit $n \to \infty$ and next $\varepsilon \to 0$ we have $|J_n - J| \to 0$. If we take a limit in (B.4), we obtain h_0 is a Nash equilibrium in the game with θ. Moreover, $h_0 \in L_{\overline{M}}$, hence $h_0 \in LNE_{\overline{M}}(\theta)$. □

Proof of Theorem 6. Let a MSNE $h^* \in L_M$ be given. For a transition probability $Q(\cdot|x-h^*(x),x)$ define a corresponding Markov operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ by

$$Tf(x) = g(x-h^*(x))\int_I f(y)\lambda(dy|x) + (1-g(x-h^*(x)))f(0).$$

Observe, the operator T is stable; hence $Q(\cdot|x-h^*(x),x)$ has a Feller property. We now show T is also quasi-compact.[32] To see that this is the case, define an operator L:

$$Lf(x) = (1-g(x-h(x)))f(0),$$

in $\mathcal{C}(I)$. Endow $\mathcal{C}(I)$ with the sup norm and denote a unit ball in $\mathcal{C}(I)$ by K. Let f be an arbitrary element from K. Note that

$$L(K) = \{(1-g(x-h(x)))f(0) : f \in K\}.$$

Clearly,

$$L(K) = \{\alpha(1-g(x-h(x))) : \alpha \in [0,1]\}$$

is the compact set. Hence, L is a compact operator. Let $f \in K$, then

$$|Tf(x) - Lf(x)| = \left|g(x-h(x))\int_I f(y)\lambda(dy|x)\right|$$

$$\leq g(x-h(x))\int_I |f(y)|\lambda(dy|x) \leq \|g\|_\infty < 1.$$

Hence, $\|T-L\| < 1$ and hence T is quasi-compact.

Finally, applying Theorem 3.3 from Futia (1982), we conclude that T is equicontinuous. Further, observe that $Q(0|S-h^*(S),S) > 0$, and $Q(0|0,0) > 0$. Therefore, Theorem 2.12 Futia (1982) shows that if an operator T is equicontinuous and Q satisfies the above mixing condition, then the Markov process induced by Q and h^* has a unique invariant distribution μ^*. Moreover, we get that from any initial $x_0 \in I$, the measure on I induced by Q and h^* converges to μ^*. □

Proof of Theorem 7. For any $x \in I_i$:

$$|A_m^n h_m^0(x) - h^*(x)| \leq |A(A_m^{n-1}h_m^0)(x_i) - A(A^{n-1}h^0)(x_i)| + |A^n h^0(x_i) - A^n h^0(x)| + |A^n h^0(x) - h^*(x)|, \tag{B.9}$$

[32] An operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ is said to be quasi-compact iff there exists a natural number n and a compact operator L such that $\|T^n - L\| < 1$.

where above follows from the definition of A_m and a triangle inequality. Further, by Theorem 3 A maps L_M into L_M, and by point (ii) in Theorem 4 we obtain

$$|A^n h^0(x_i) - A^n h^0(x)| + |A^n h^0(x) - h^*(x)| \le Md_m + M_A(1-\tau^{r^n}). \tag{B.10}$$

Next, observe in the sup norm:

$$\|A(A_m^{n-1}h_m^0) - A(A^{n-1}h^0)\| \le \sup_{k=1,2,\ldots} \|A^k(A_m^{n-1}h_m^0) - A^k(A^{n-1}h^0)\| \le \|A_m^{n-1}h_m^0 - A^{n-1}h^0\|,$$

where the first inequality follows from the definition of the sup, operator, and the second follows by point 9 in Leader (1982) theorem. Specifically we can apply the main theorem in Leader (1982) as A is continuous, uniformly contractive and L_M compact. As a result we conclude that, there exists a number k for iterations of A giving smaller distance than the sup, distance between any two starting points, e.g. $A_m^{n-1}h_m^0$ and $A^{n-1}h^0$, applying the above n times, we obtain

$$\begin{aligned}\|A(A_m^{n-1}h_m^0) - A(A^{n-1}h^0)\| &\le \|A_m^{n-1}h_m^0 - A^{n-1}h^0\| \\ &\le Md_m + \|A(A_m^{n-2}h_m^0) - A(A^{n-2}h^0)\| \le \cdots \le (n-1)Md_m + \|h_m^0 - h^0\| = nMd_m.\end{aligned} \tag{B.11}$$

Combining the expressions (B.9)–(B.11) we obtain

$$\|A_m^n h_m^0 - h^*\| \le \|A(A_m^{n-1}h_m^0) - A(A^{n-1}h^0)\| + d_m + \|A^n h^0 - h^*\| \le (n+1)Md_m + M_A(1-\tau^{r^n}). \tag{B.12}$$

The first assertion follows from (B.12) by taking limits with $m \to \infty$, and next with $n \to \infty$. □

Proof of Lemma 8. Observe that $c_1^* = \overline{\theta}$ and that $c_T^* = A^{T-1}\overline{\theta}$. Since conditions in Theorem 9 are satisfied (via relation (9)) we obtain $\lim_{T\to\infty} A^{T-1}\overline{\theta} = h^*$ uniformly. □

Proof of Proposition 1. Let $h_1, h_2 \in \mathcal{C}(I)^\circ$ and $0 < t < 1$ be given. For simplicity, for a given $x \in I$ denote $C(h_1, h_2)(x) = \tilde{C}$ and $C(th_1, t^{-1}h_2)(x) = \tilde{C}_t$. Observe $\tilde{C} > \tilde{C}_t$. From the definition of $\tilde{C}$ and $\tilde{C}_t$ we obtain

$$\gamma\tilde{C}^{\gamma-1} + \alpha\tilde{C}^{\alpha-1}g(x-\tilde{C})\int_I h_1^\beta(y)\lambda(dy|x) - \tilde{C}^\alpha g'(x-\tilde{C})\int_I h_2^\beta(y)\lambda(dy|x) = 0,$$

Similarly, for $\tilde{C}_t$, we have

$$\gamma\tilde{C}_t^{\gamma-1} + \alpha\tilde{C}_t^{\alpha-1}g(x-\tilde{C}_t)\int_I t^\beta h_1^\beta(y)\lambda(dy|x) - \tilde{C}_t^\alpha g'(x-\tilde{C}_t)\int_I t^{-\beta}h_2^\beta(y)\lambda(dy|x) = 0.$$

Solving for $\int_I h_2^\beta(y)\lambda(dy|x)$ from the latter equation, and substituting the result into the former, we have

$$t^\beta \frac{\tilde{C}^\alpha g'(x-\tilde{C})}{\tilde{C}_t^\alpha g'(x-\tilde{C}_t)}\left[\gamma\tilde{C}_t^{\gamma-1} + \alpha\tilde{C}_t^{\alpha-1}g(x-\tilde{C}_t)\int_I t^\beta h_1^\beta(y)\lambda(dy|x)\right] = \left[\gamma\tilde{C}^{\gamma-1} + \alpha\tilde{C}^{\alpha-1}g(x-\tilde{C})\int_I h_1^\beta(y)\lambda(dy|x)\right].$$

As g is strictly concave, and $0 < \alpha < 1$, we have $\tilde{C}^\alpha g'(x-\tilde{C})/\tilde{C}_t^\alpha g'(x-\tilde{C}_t) > 1$; hence

$$t^\beta\left[\gamma\tilde{C}_t^{\gamma-1} + \alpha\tilde{C}_t^{\alpha-1}g(x-\tilde{C}_t)\int_I t^\beta h_1^\beta(y)\lambda(dy|x)\right] < \left[\gamma\tilde{C}^{\gamma-1} + \alpha\tilde{C}^{\alpha-1}g(x-\tilde{C})\int_I h_1^\beta(y)\lambda(dy|x)\right]. \tag{B.13}$$

We now show by contradiction that $\tilde{C}_t \ge t^{2\beta/(1-\alpha)}\tilde{C}$. Assume $\exists \tilde{x} \in I, \tilde{x} > 0$ such that $\tilde{C}_t(\tilde{x}) < t^{2\beta/(1-\alpha)}\tilde{C}(\tilde{x})$. It follows that $t^{2\beta}\tilde{C}_t^{\alpha-1}(\tilde{x}) > \tilde{C}^{\alpha-1}(\tilde{x})$. Moreover, as g and h_1 are nonnegative, and $g(\tilde{x}-\tilde{C}_t(\tilde{x})) > g(\tilde{x}-\tilde{C}(\tilde{x}))$, we have

$$\alpha t^\beta\tilde{C}_t^{\alpha-1}(\tilde{x})g(\tilde{x}-\tilde{C}_t(\tilde{x}))\int_I t^\beta h_1^\beta(y)\lambda(dy|x) \ge \alpha\tilde{C}^{\alpha-1}(\tilde{x})g(\tilde{x}-\tilde{C}(\tilde{x}))\int_I h_1^\beta(y)\lambda(dy|x). \tag{B.14}$$

Since $\gamma \le \alpha$, we have $\beta/(1-\gamma) \le \beta/(1-\alpha)$. With $0 < t < 1$, we obtain $t^{2\beta/(1-\alpha)}\tilde{C}(\tilde{x}) < t^{\beta/(1-\alpha)}\tilde{C}(\tilde{x}) \le t^{\beta/(1-\gamma)}\tilde{C}(\tilde{x})$. Combining this result with the assumption, we have: $\tilde{C}_t(\tilde{x}) < t^{\beta/(1-\gamma)}\tilde{C}(\tilde{x})$. Which leads to the following:

$$t^\beta\tilde{C}_t^{\gamma-1}(\tilde{x}) > \tilde{C}^{\gamma-1}(\tilde{x}). \tag{B.15}$$

Adding inequalities (B.14) and (B.15), we obtain a contradiction (B.13) at $\tilde{x}$. Hence $\tilde{C}_t \ge t^{2\beta/(1-\alpha)}\tilde{C}$.

By assumption $2\beta + \alpha < 1$ hence $r := 2\beta/(1-\alpha) < 1$; therefore, the hypotheses of condition (A.4) in a Theorem 10 is satisfied. Since $\mathcal{C}(I)$ is a subset of a normal solid cone in a real Banach space, and C mixed monotone with $C(\cdot,\cdot) \ne \underline{\theta}$, we conclude from Theorem 10 (see Appendix) the existence of a unique fixed point of C on $\mathcal{C}(I)$. □

References

Abreu, D., Pearce, D., Stacchetti, E., 1990. Toward a theory of discounted repeated games with imperfect monitoring. Econometrica 58 (5), 1041–1063.
Alvarez, F., 1999. Social mobility: the Barro–Becker children meet the Laitner–Loury dynasties. Review of Economic Studies 2 (1), 65–103.
Amann, H., 1977. Order Structures and Fixed Points, SAFA 2, ATTI del 2. Seminario di Analisi Funzionale e Applicazioni, MS.
Amir, R., 1996a. Continuous stochastic games of capital accumulation with convex transitions. Games and Economic Behavior 15 (2), 111–131.
Amir, R., 1996b. Strategic intergenerational bequests with stochastic convex production. Economic Theory 8 (2), 367–376.
Amir, R., 2002. Complementarity and diagonal dominance in discounted stochastic games. Annals of Operations Research 114, 39–56.
Athey, S., Atkeson, A., Kehoe, P.J., 2005. The optimal degree of discretion in monetary policy. Econometrica 73 (5), 1431–1475.
Atkeson, A., 1991. International lending with moral hazard and risk of repudiation. Econometrica 59 (4), 1069–1089.

Balbus, Ł., Nowak, A.S., 2004. Construction of Nash equilibria in symmetric stochastic games of capital accumulation. Mathematical Methods Operations Research 60 (2), 267–277.
Balbus, Ł., Nowak, A.S., 2008. Existence of perfect equilibria in a class of multigenerational stochastic games of capital accumulation. Automatica 44 (6), 1471–1479.
Balbus, Ł., Reffett, K., Woźny, Ł., 2012a. Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities, MS.
Balbus, Ł., Reffett, K., Woźny, Ł., 2012b. On Constructive Monotone Methods for Studying Time-Consistent Markov Policies of Quasi-Hyperbolic Consumers Under Uncertainty, MS.
Balbus, Ł., Reffett, K., Woźny, Ł., 2012c. Stationary Markovian equilibrium in altruistic stochastic OLG models with limited commitment. Journal of Mathematical Economics 48 (2), 115–132.
Beer, G., 1980. The approximation of upper semicontinuous multifunction by step multifunction. Pacific Journal of Mathematics 87 (1), 11–19.
Bernheim, D., Ray, D., 1987. Economic growth with intergenerational altruism. Review of Economic Studies 54 (2), 227–242.
Bernheim, D., Ray, D., Yeltekin, S., 1999. Self-Control, Savings, and the Low-Asset Trap, MS.
Bertsekas, D., 1975. Convergence of discretization procedures in dynamic programming. IEEE Transactions on Automatic Control 20 (3), 415–419.
Cid-Araújo, J., 2004. The uniqueness of fixed points for decreasing operators. Applied Mathematics Letters 17 (7), 861–866.
Coleman, Wilbur John I., 1991. Equilibrium in a production economy with an income tax. Econometrica 59 (4), 1091–1104.
Curtat, L., 1996. Markov equilibria of stochastic games with complementarities. Games and Economic Behavior 17 (2), 177–199.
Datta, M., Reffett, K., 2006. Isotone recursive methods: the case of homogeneous agents. In: Kazuo, N., Dana, R., Le Van, C., Mitra, T. (Eds.), Handbook on Optimal Growth, Springer, Berlin, Heidelberg.
Feng, Z., Miao, J., Peralta-Alva, A., Santos, M., 2009. Numerical Simulation of Nonoptimal Dynamic Equilibrium Models, MS.
Fox, B., 1973. Discretizing dynamic programs. Journal of Optimization Theory and Applications 11 (3), 228–234.
Futia, C.A., 1982. Invariant distributions and the limiting behavior of Markovian economic models. Econometrica 50 (2), 377–408.
Guo, D., Cho, Y.J., Zhu, J., 2004. Partial Ordering Methods in Nonlinear Problems. Nova Science Publishers, Inc., New York.
Guo, D., Lakshmikantham, V., 1988. Nonlinear Problems in Abstract Cones. Academic Press, Inc., San Diego.
Hewitt, E., Stromberg, K., 1965. Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable. Springer, Berlin, Heidelberg.
Hinderer, K., 2005. Lipschitz continuity of value functions in Markovian decision processes. Mathematical Methods Operations Research 62 (1), 3–22.
Hitzler, P., Seda, A.K., 2001. A "converse" of the Banach contraction mapping theorem. Journal of Electrical Engineering 52 (10/s), 3–6.
Horst, U., 2005. Stationary equilibria in discounted stochastic games with weakly interacting players. Games and Economic Behavior 51 (1), 83–108.
Janos, L., 1967. A converse of Banach's contraction theorem. Proceedings of the American Mathematical Society 18 (2), 287–289.
Judd, K.L., Yeltekin, S., Conklin, J., 2003. Computing supergame equilibria. Econometrica 71 (4), 1239–1254.
Kelley, J., 1955. General Topology. Van Nostrand, New York.
Krasnosel'skiĭ, M., Zabreĭko, P., 1984. Geometrical Methods of Nonlinear Analysis. Springer, Berlin, Heidelberg.
Kuratowski, K., 1966. Topology, vol. 1. Academic Press, New York.
Kydland, F., Prescott, E., 1977. Rules rather than discretion: the inconsistency of optimal plans. Journal of Political Economy 85 (3), 473–491.
Kydland, F., Prescott, E., 1980. Dynamic optimal taxation, rational expectations and optimal control. Journal of Economic Dynamics and Control 2 (1), 79–91.
Laitner, J., 1979a. Household bequest behaviour and the national distribution of wealth. Review of Economic Studies 46 (3), 467–483.
Laitner, J., 1979b. Household bequests, perfect expectations, and the national distribution of wealth. Econometrica 47 (5), 1175–1193.
Laitner, J., 2002. Wealth inequality and altruistic bequests. American Economic Review 92 (2), 270–273.
Leader, S., 1982. Uniformly contractive fixed points in compact metric spaces. Proceedings of the American Mathematical Society 86 (1), 153–158.
Leininger, W., 1986. The existence of perfect equilibria in model of growth with altruism between generations. Review of Economic Studies 53 (3), 349–368.
Loury, G., 1981. Intergenerational transfers and the distribution of earnings. Econometrica 49 (4), 843–867.
Marcet, A., Marimon, R., 1998. Recursive Contracts, MS.
Messner, M., Pavoni, N., 2004. On the Recursive Saddle Point Method. MS.
Mirman, L., Morand, O., Reffett, K., 2008. A qualitative approach to Markovian equilibrium in infinite horizon economies with capital. Journal of Economic Theory 139 (1), 75–98.
Nowak, A.S., 2003. On a new class of nonzero-sum discounted stochastic games having stationary Nash equilibrium points. International Journal of Game Theory 32 (1), 121–132.
Nowak, A.S., 2006. On perfect equilibria in stochastic models of growth with intergenerational altruism. Economic Theory 28 (1), 73–83.
Nowak, A.S., 2007. On stochastic games in economics. Mathematical Methods of Operations Research 66 (3), 513–530.
Peleg, B., Yaari, M.E., 1973. On the existence of a consistent course of action when tastes are changing. Review of Economic Studies 40 (3), 391–401.
Peralta-Alva, A., Santos, M., 2010. Problems in the numerical simulation of models with heterogeneous agents and economic distortions. Journal of the European Economic Association 8 (2–3), 617–625.
Phelan, C., Stacchetti, E., 2001. Sequantial equilibria in a Ramsey tax model. Econometrica 69 (6), 1491–1518.
Phelps, E., Pollak, R., 1968. On second best national savings and game equilibrium growth. Review of Economic Studies 35 (2), 195–199.
Rustichini, A., 1998. Dynamic programming solution of incentive constrained problems. Journal of Economic Theory 78 (2), 329–354.
Santos, M., Peralta-Alva, A., 2005. Accuracy of simulations for stochastic dynamic models. Econometrica 73 (6), 1939–1976.
Sundaram, R.K., 1989. Perfect equilibrium in non-randomized strategies in a class of symmetric dynamic games. Journal of Economic Theory 47 (1), 153–177.