On collective intertemporal choice, time-consistent decision rules and altruism*

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Abstract

We consider a dynamic decision problem of a collective-household with heterogeneous discount factors. We prove existence a time-consistent decision rule in Markovian policies. In fact, decision rules we consider are characterized by monotone and Lipschitz continuous investments policies. We provide sufficient conditions for the validity of the (generalized) first-order approach. Finally, we propose a novel method to approximate the constructed decision rule by a sequence of equilibria of bequest games with paternalistic altruism.

Keywords: heterogeneous discounting, collective intertemporal choice, paternalistic altruism, bequest games, collective household models, time-consistent policy rules, approximation
JEL classification C62, C73.

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1 Introduction

Over the last decades there has been a growing interest in the characterization of solutions to dynamic *collective* decision problems (see Marglin (1963) and Feldstein (1964) for early references). Indeed, evaluating public investments, policy proposals, conducting environmental cost-benefit analysis or simply making collective household decisions inevitably concerns a group of individuals or the whole society. A central challenge in such collective decision problems is the heterogeneity of individual preferences, most notably with respect to evaluation of future consumption streams. Such problems were recently analyzed by Becker (2012), Chiappori and Mazzocco (2017), Feng and Ke (2018), Millner and Heal (2018) or Ebert, Wei, and Zhou (2020) in various theoretical, experimental or empirical, settings.

Gollier and Zeckhauser (2005) or, later, Zuber (2011) and Jackson and Yariv (2015), demonstrated that the collective or non-dictatorial group preferences over sequences of consumptions are typically *time-inconsistent*. As a consequence, a planned consumption path may not be executed in the future unless some form of commitment (devices) is available. Notably, this is so even when each individual dynamic utility is time consistent, but agents differ in their discount factors only. This finding not only questions the validity of the representative household assumption in dynamic macro-models but also requires to answer a more fundamental problem of defining, characterizing and computing the relevant decision rules in such settings.

In view of the above observation, two dominant solution strategies have been proposed. First, economists may focus on Pareto-*optimal* allocations. Such an approach was advocated and analyzed by Gollier and Zeckhauser (2005) or Alcala (2018) to name just a few recent contributions. Indeed, following the approach of Lucas and Stokey (1984) with appropriately chosen Negishi weights a (set) of sequential solutions can be characterized. Alternatively, economists studied *time-consistent* solutions under these inherently time-inconsistent preferences.[1] Specifically, when lacking suitable commitment devices a

[1] Both approaches suffer from well know predicaments. Pareto-optimal solutions typically require infinite memory and are not stationary nor Markovian (on the minimal state space), while time-consistent solutions, upon existence, are typically only constrained optimal.

form of intergenerational equilibrium is imposed so that the future generations do not have incentives to deviate from the planned consumption path.[2] Following that approach Drugeon and Wigniolle (2016, 2020) proposed a Markov stationary solution concept and characterized its implied allocations using a first order approach. While they did not complete a general existence argument, they provided a comprehensive account for various examples involving logarithmic or CIES preferences and Cobb-Douglas technologies. Generally, however, it is well known from the literature on various forms of (quasi-)hyperbolic discounting that such first order characterization is often not available due to missing continuity or concavity of the continuation values (see e.g. Harris and Laibson (2001) or Chatterjee and Eyigungor (2016) more recently). Moreover, as demonstrated by Caplin and Leahy (2006), *e.g.*, a time-consistent solution may be non-existent in models with changing preferences. This is particularly challenging when restricting attention to a class of stationary strategies. To circumvent this problem, restore existence and validity of the first order-characterization some forms of noise or stochastic transitions were introduced to smooth the (expected) continuation values (see, *e.g.*, Harris and Laibson (2001) or Balbus, Reffett, and Woźny (2015)).

In this paper we also follow that approach and consider a stochastic environment with transitions characterized by the convexity of the distribution functions. Under this condition, in particular, we prove the existence of a time-consistent decision rule in Markovian policies (section 2) under standard assumptions on preferences. This result is new and generalizes earlier existence results known for a class of bequest games or (quasi)-hyperbolic discounting. As a matter of fact, the decision rules we consider are characterized by monotone and Lipschitz continuous aggregate investment policies and the same concerns individual consumption policies. We next provide sufficient conditions for the validity of the (generalized) first-order approach[3] (section 3). When doing so we extend the earlier results of Drugeon and Wigniolle (2020) on heterogeneous agents economies and Harris and Laibson (2001) and Balbus, Reffett, and Woźny (2018) on related quasi-hyperbolic

[2] A separate line of research discussed also possible decentralization mechanisms of such equilibrium allocations (see, *e.g.*, Herings and Rohde (2006), Luttmer and Mariotti (2007) or Dziewulski (2015)).

[3] See Harris and Laibson (2001) for a classic reference to generalized first-order equations for a quasi-hyperbolic discounting problem.

discounting. This extension is significant as the collective decision problem we consider does not possess any stationary structure. Finally, we propose a novel method to approximate the constructed stationary decision rule by a sequence of equilibria of bequest games with paternalistic altruism (section 4). We thus propose a method to approximate the sequence of first order conditions of the collective household model. This is related to recent contribution of Galperti and Strulovici (2017) on non-paternalistic altruism and extends the results of Balbus, Reffett, and Woźny (2020) on single-agent behavioral discounting models to collective decision problems.

2 Definition and existence

Time is discrete and horizon infinite. Consider a dynamic maximization problem of a collective household with two individuals.[4] Instantaneous utility functions of both individuals are given by u^1 and u^2, and we assume both discount the future utility streams exponentially with discount factors given by δ_1 and δ_2 respectively. The first individual is more patient with $1 > \delta_1 > \delta_2 > 0$. Without loss of generality, the weights of both utilities in the collective household preferences are 1 and $\eta > 0$. Suppose that the initial state[5] is $s \in [0, \bar{S}] := S \subset \mathbb{R}$. We interpret it as an asset holding or an aggregate production level. The utility of the collective household is then the following:

$$\mathcal{U}_s\left((c_{1,t}, c_{2,t})_{t=0}^{\infty}\right) := E_s \left\{ \sum_{t=0}^{\infty} (\delta_1)^t u^1(c_{1,t}) + \eta \sum_{t=0}^{\infty} (\delta_2)^t u^2(c_{2,t}) \right\},$$

where $c_{i,t} \geq 0$ is the i-th individual consumption in period t. Consumption choices are constrained by output / asset level s_t, i.e. $c_{1,t} + c_{2,t} \leq s_t$, and we assume that s_t is a Markov chain controlled by $(c_{1,t}, c_{2,t})_{t=0}^{\infty}$, such that state s_{t+1} is drawn from distribution $Q(\cdot|s_t - c_{1,t} - c_{2,t})$. Here E_s denotes an expectation operator induced by the random vector $(s_t, c_{1,t}, c_{t,2})_{t=0}^{\infty}$ with an initial state given by $s \in S$.

[4] Generalizations to any finite number of individuals are straightforward.

[5] Here we assume bounded state space and necessarily bounded reward space. The generalization of our approach to unbounded state space and unbounded above utility functions can be developed using results on local contractions (Rincon-Zapatero and Rodriguez-Palmero, 2003).

We now focus on a dynastic representation of this collective household problem. That is, suppose that the initial generation (date-0) with state s has a utility $\mathcal{U}_s\left((c_{1,t}, c_{2,t})_{t=0}^{\infty}\right)$ while generation's τ utility is given by: $\mathcal{U}_{s_\tau}\left((c_{1,t}, c_{2,t})_{t=\tau}^{\infty}\right)$. Preferences of a sequence of generations are hence *time-invariant, non-stationary and time-inconsistent*:

$$(1) \quad \mathcal{U}_s\left((c_{1,t}, c_{2,t})_{t=0}^{\infty}\right) = E_s \sum_{t=0}^{\infty} (\delta_1)^t \left(u^1(c_{1,t}) + \eta \left(\frac{\delta_2}{\delta_1}\right)^t u^2(c_{2,t}) \right).$$

Indeed, it is clear from the above formulation that within-period aggregate preferences are non-stationary with the weight of the impatient individual converging to zero with t. As a result, optimal solutions to (1) are time-inconsistent as they are based on the promise to increase the relative weight of the patient consumer in the future but since the problem is time-invariant such promise cannot be executed unless some form of commitment is imposed.[6]

For this reason, from now on we seek a time-consistent solution to such a time-inconsistent problem. In particular we concentrate on solutions in stationary Markovian strategies, *i.e.*, where each period consumption and investment decisions are (the same) functions of the current period state only. Such solutions are of particular interests, whenever future generations have short memory and are allowed to modify the planned consumption / allocation paths or re-optimize them according to their current preferences. Indeed, when lacking appropriate commitment devices, the current generation does not have tools to enforce the planned continuation path and a form of intergenerational equilibrium or time-consistency is required to ensure that the plan is implemented or actually followed.

Characterizing time-consistent solutions in such an environment is non-trivial. In particular, the available methods known from the literature on related quasi-hyperbolic discounting are not applicable here as the sequence of aggregate period preferences do not possess any stationary structure. More explicitly, existence results in stationary strategies in models of *quasi-hyperbolic* discounting heavily use the fact that (i) preferences

[6] We refer the reader to Nowak (2010) who studies a related dynamic, noncooperative game with internally cooperating generations.

non-stationarity is summarized by two parameters (β and δ) and (ii) the beliefs of each generation on its preferences from the next period are stationary. Such a structure is unequivocally not available in the collective household problem that is addressed in this paper.[7]

To proceed, we start with some typical assumptions on preferences.

Assumption 1. *Each $u^i : S \to \mathbb{R}$ is continuous, strictly increasing and strictly concave.*

Generation 0 decides on a total consumption level $c_0 \in [0, s]$ and allocation $(c_{1,0}, c_{2,0})$ such that $c_0 = c_{1,0} + c_{2,0}$. Then

$$u^1(c_{1,0}) + \eta u^2(c_{2,0}) \geq u^1(c_1) + \eta u^2(c_2) \quad \text{for any } c_1, c_2 \geq 0 \text{ such that } c_1 + c_2 = c_0.$$

We start with a characterization of the optimal division of c_0 among both consumers.

Lemma 1. *Assume 1. Then $c_{1,0}$ and $c_{2,0}$ are uniquely determined. Moreover, they are increasing and Lipschitz continuous functions of c_0.*

Proof. It follows from Lemma 10 by taking $f_1 = u^1$ and $f_2 = \eta u^2$. □

By Assumption 1 and Lemma 1, generation 0 allocations can be hence determined by $c_{1,0} = \beta(c_0)$ and $c_{2,0} = \gamma(c_0)$, where β and γ are both increasing and Lipschitz continuous functions. In fact, in the stationary solution, all generations t would follow the same policy and $c_{1,t} = \beta(c_t)$, $c_{2,t} = \gamma(c_t)$ for selected $c_t \in [0, s_t]$. Anticipating on that solution, we may alternatively define the date-0 utility as follows:

$$(2) \quad \mathscr{U}_s\left((c_t)_{t=0}^{\infty}\right) = \mathcal{U}_s\left((\beta(c_t), \gamma(c_t))_{t=0}^{\infty}\right) = E_s \sum_{t=0}^{\infty} (\delta_1)^t \left(u^1(\beta(c_t)) + \eta \left(\frac{\delta_2}{\delta_1}\right)^t u^2(\gamma(c_t)) \right).$$

For $c \in S$, let us then define

$$(3) \quad u_t(c) := u^1(\beta(c)) + \eta \left(\frac{\delta_2}{\delta_1}\right)^t u^2(\gamma(c)).$$

[7] Some progress has been made with respect class of models with generalized quasi-hyperbolic discounting and upper-semicontinuous strategies (see Balbus, Reffett, and Woźny (2020)).

In the above, u_t is hence the instantaneous utility each generation assigns to the consumption t-periods ahead, obtained according to anticipated decision rules β and γ of the following generations. We proceed with a characterization of u_t.

Lemma 2. *Assume 1. The function u_0 is continuous, strictly increasing and strictly concave; for any $t \geq 1$, the function u_t is continuous and strictly increasing.*

Proof. For u_0 the result follows from Lemma 10. For u_t it follows from the definition of γ and β. □

Suppose now that the aggregate consumption plan is $h : S \mapsto S$, where h is Borel measurable and $h(s) \in [0, s]$. That is, after observing the state s_t, the current generation t chooses $c_t = h(s_t)$ and allocates $c_{1,t} = \beta(h(s_t))$ and $c_{2,t} = \gamma(h(s_t))$. Put

$$V\left(h\right)\left(s\right) = E_s \sum_{t=0}^{\infty} (\delta_1)^t u_t\left(h(s_t)\right).$$

Let U be the continuation value, *i.e.*,

$$U\left(h\right)\left(s\right) := E_s \sum_{t=1}^{\infty} (\delta_1)^{t-1} u_t\left(h(s_t)\right).$$

We can then write

$$V\left(h\right)\left(s\right) = u_0(h(s)) + \delta_1 \int_S U\left(h\right)(s')Q(ds'|s - h(s)),$$

where the discounting of this criterion is based upon the most patient agent and its lack of standard recursivity results from the discrepancy between U and V. With that notation in mind, we are now ready to introduce our equilibrium concept.

Definition 1. The measurable profile $h^* : S \mapsto S$ is a Stationary Markov Perfect Equilibrium (SMPE) if, for any $s \in S$,

$$V(h^*)(s) = \max_{c \in [0,s]} \left\{ u_0(c) + \delta_1 \int U(h^*)(s')Q(ds'|s - c) \right\}.$$

That is, *the SMPE policy h^* is the best response to the value $U(h^*)$ that is itself generated by the anticipated use of that same policy h^* by the whole sequence of future generations.* This is a strong equilibrium concept that implies time-consistency in our setting and thus allows to sustain investment $s - h^*(s)$ and consumptions $\beta(h^*(s))$ and $\gamma(h^*(s))$ in any period.

We will now proceed to verify the existence of such a SMPE. As this is typically the case in the literature, our construction is based on a fixed point argument. When doing so one first needs to choose an appropriate (compact and convex) function space that is mapped (by the best-response) into itself. It has been known, however, since at least Leininger (1986) that the selection of such a domain is a difficult task in a class of dynamic games due to the "vicious circle" of the strategy space. That problem is also present here. Indeed and more explicitly, whenever the consecutive generations use, *e.g.*, concave policies it is generally not clear whether the best response maps back into that space of concave policies. In the current collective household problem, with a lack of the principle of optimality, such a condition is even more restrictive, as this is both aggregate consumption and investments that would need to be concave. The same difficulty emerges for best responses to continuous policies of the following generations. As the continuation value function is not generally concave (see lemma 2), it is difficult to ensure that the best response is indeed continuous as a function of the current state. While some progress has been recently made in the literature by restricting attention to a class of increasing and upper (or lower) semi-continuous policies endowed with the weak topology (see, *e.g.*, Balbus, Jaśkiewicz, and Nowak (2015)), the following example makes clear that such an approach is of limited use[8] in the *deterministic* collective household problem.

Example 1. Let $S = [0, \bar{s}]$, with large enough $\bar{s} > 1$. For any $s \in S$ and bundle $(c_1, c_2) \in S^2$ such that $c_1 + c_2 \in [0, s]$ let $u^1(c_1) := \sqrt{c_1}/\sqrt{2}$ and $u^2(c_2) = \sqrt{c_2}/\sqrt{2}$.

[8]Observe that example 1 is not *per se* a counterexample to existence of SMPE. It rather illustrates how the general approach using the fixed point argument may fail when applied to the case of a deterministic transition.

Furthermore, $\eta = 1$, $\delta_1 = 4/5$, $\delta_2 = 1/5$. Clearly

$$u_0(c) = \max_{c_1,c_2 \geq 0 \text{ s.t. } c_1+c_2 \leq c} \{u(c_1) + \eta u(c_2)\} = \sqrt{c},$$

and $\beta(c) = \gamma(c) = c/2$. Let deterministic transition by given by $F(i) = \sqrt{i}$. Then,

$$u_t(c) = u^1(\beta(c)) + \eta \left(\frac{\delta_2}{\delta_1}\right)^t u^2(\gamma(c)) = \frac{1}{2}\sqrt{c}\left(1 + \left(\frac{1}{4}\right)^t\right).$$

Consider a consumption policy $h(s) = s$ for $s < 1$ and $h(s) = 0$ for $s \geq 1$. Observe that this corresponds to an increasing and upper semi-continuous investment policy. We show that the first generation has no best response to h, when the current state is $s = 17$. Indeed, if $i < 1$ then the next state is $\sqrt{i}$, still less than 1, hence the next generation consumes everything and the first generation payoff is

$$f(i) := u_0(17 - i) + \delta_1 u_1(F(i)) = \sqrt{17 - i} + \frac{1}{2}\sqrt[4]{i}.$$

If $i \geq 1$, the next state is $\sqrt{i}$, still greater than 1, and the same goes for all future generations, each consuming nothing. In such case the payoff is

$$f(i) := u(17 - i) = \sqrt{17 - i}.$$

Observe that, for $i < 1$,

$$f'(i) = -\frac{1}{2\sqrt{17 - i}} + \frac{1}{8\sqrt[4]{i^3}} > -\frac{1}{2\sqrt{16}} + \frac{1}{8} = 0,$$

hence f increases on $[0, 1)$ before reaching the limit

$$\lim_{i \uparrow 1} f(i) = \sqrt{16} + \frac{1}{2} = \frac{9}{2}.$$

But $f(1) = \sqrt{16} = 4 < \frac{9}{2}$ and f decreases on $[1, \bar{s}]$. Hence there is no best response to such h and the fixed point approach to equilibrium construction cannot be applied.

A similar (counter)example can be constructed whenever the next generations use an (increasing) investment policy that is lower semi-continuous. Indeed, let the consumption policy by given by $h(s) = s$ for all $s \leq 1$ and $h(s) = s/2$ for $s > 1$. Let $s = 4$. If $i \leq 1$ then $\sqrt{i} \leq 1$ as well and the next generation consumes everything. The resulting payoff is:

$$f(i) := u_0(s-i) + \delta_1 u_1(F(i)) = \sqrt{4-i} + \frac{1}{2}\sqrt[4]{i}.$$

If $i > 1$ then $\sqrt{i} > 1$ and the next and second generation invests $\sqrt{i}/2$ and consumes the same amount; the third generation however inherits $\sqrt{\sqrt{i}/2} \leq 1$ and hence consumes everything. Hence the current generation payoff is:

$$f(i) = \sqrt{4-i} + \frac{1}{2}\sqrt{\frac{1}{2}\sqrt{i}} + \frac{1}{2}\sqrt{\sqrt{\frac{1}{2}\sqrt{i}}}\left(\left(\frac{4}{5}\right)^2 + \left(\frac{1}{5}\right)^2\right).$$

Again, it can be easily showed (see appendix for a graphical exposition) that there is no argument maximizing f on $[0, 4]$.

In the view of the above discussion and (counter)example, we now present a different set of assumptions that allows to circumvent the illustrated problems. We start by considering a function space CM of continuous and monotone aggregate consumption policies such that investment is also increasing:

$$CM := \Big\{ h : S \to S \mid h \text{ is continuous, increasing and s.t. } s \to s - h(s) \text{ is increasing} \Big\}.$$

In fact, consumption policies in CM are Lipschitz continuous.

Consider also the $\mathscr{V}$ space of implied values:

$$\mathscr{V} := \left\{ v : S \to \mathbb{R} \mid v \text{ is continuous, increasing, with } \quad 0 < v < \frac{u^1(\bar{S}) + \eta u^2(\bar{S})}{1-\delta_1} \right\}.$$

Endow then both CM and $\mathscr{V}$ with the sup-norm topology, *i.e.*, $\|f\|_\infty := \sup_{s \in S} |f(s)|$.

Key assumptions are now going to be introduced on the stochastic transition Q.

Assumption 2. *For any $v \in \mathscr{V}$, the function $i \to \int_S v(s')Q(ds'|i)$ is continuous, increasing and concave.*

Assumption 2 builds from three distinct properties on Q. A Feller property is first imposed on Q so that the expected value is continuous. Second, stochastic monotonicity on Q guarantees that this expected value is increasing in investment i. And third, stochastic concavity implies that this expected value is concave as a function of investment.[9] All three conditions are required to hold relative to $v \in \mathscr{V}$.

We shall now clarify the status of this stochastic convexity assumption, first by a remark that illustrates how if may fail to be satisfied for a deterministic transition and then by giving an explicit example of a stochastic transition satisfying this assumption.

Remark 1 (Necessity of the stochastic transition). Observe that the stochastic convexity of Q in Assumption 2 rules out deterministic transitions. Indeed, it is straightforward to product counterexamples of expected values that are not concave in i relative to values in $\mathscr{V}$ if Q is deterministic. Moreover, as showed in Example 1 existence of SMPE in the collective household model under deterministic transition is generally not guaranteed.

Example 2. An example of transition Q satisfying Assumption 2 is available by simply considering a weighted average of two measures λ_2 and λ_1, namely $Q(\cdot|i) := g(i)\lambda_2(\cdot) + (1 - g(i))\lambda_1(\cdot)$, where λ_2 and λ_1 are measures on S, with λ_2 first order stochastically dominating λ_1 and a function $g : S \to [0, 1]$ which is continuous, increasing and concave.

For $v \in \mathscr{V}$, let the operator $T_t^h(v)(s)$ be defined as follows:

$$T_t^h(v)(s) = u_t(h(s)) + \delta_1 \int_S v(s')Q(ds'|s - h(s)).$$

For the sequence $v = (v_t)_{t=1}^{\infty} \in \mathscr{V}^{\infty}$, we further define $T^h(v)(s) = (T_t^h(v_{t+1})(s))_{t=1}^{\infty}$. Endow $\mathscr{V}^{\infty}$ with the following norm:

$$\|v\|^{\kappa} = \sum_{t=1}^{\infty} \frac{\|v_t\|_{\infty}}{\kappa^{t-1}},$$

[9] This last condition on Q was introduced in, *e.g.*, moral hazard models by Rogerson (1985) or stochastic games by Amir (1996).

where $\kappa \in (1, 1/\delta_1)$.

It is first easy to see that, by Assumptions 1 and 2, T^h is a contraction in this norm and has a unique fixed point $v^h = (v_t^h)_{t=1}^\infty$. This is summarized in the following lemma.

Lemma 3. *Assume 1 and 2. For any $h \in CM$, $T^h : \mathscr{V}^\infty \mapsto \mathscr{V}^\infty$, and it is a contraction mapping with a $\delta_1\kappa$-contraction constant. As a result, T^h has a unique fixed point $v^h = (v_t^h)_{t=1}^\infty$.*

Moreover, $U(h)(s) = v_1^h(s)$. From the definition of $T_t^h(v)(s)$, an extra continuity remains to be completed:

Lemma 4. *Assume 1 and 2. The mapping $(h, v) \in CM \times \mathscr{V}^\infty \mapsto T^h(v)$ is continuous.*

Parallelly verifying the continuity of U:

Lemma 5. *Assume 1 and 2. The operator $U : CM \mapsto \mathscr{V}$ is continuous.*

Finally, and from the Arzela-Ascoli Theorem (see chapter 7 in Kelley (1991)), it is clear that:

Lemma 6. *CM is compact in the sup norm.*

Define then, on CM, a *best response* BR operator by

$$BR(h)(s) := \arg\max_{c \in [0,s]} \left\{ u_0(c) + \delta_1 \int_S U(h)(s') Q(ds'|s - c) \right\}.$$

By Lemma 2 and under Assumption 2, $BR(h)$ is well defined for any h. We first check, whether it maps the set CM into itself.

Lemma 7. *Assume 1 and 2. Then $BR : CM \to CM$.*

We are now ready to state our main existence result:

Theorem 1. *Assume 1 and 2. There exists a SMPE, that is there exists $h^* \in CM$ such that $h^* = BR(h^*)$.*

We now characterize stochastic steady states, *i.e.,* invariant distributions implied by transition Q and equilibrium policy h^* (and related $\beta(h^*(\cdot))$ and $\gamma(h^*(\cdot))$. Let $h^* \in CM$ be a SMPE and consider the process $(s_t)_t$ such that $s_0 = s$, and for all $t \geq 0$, s_{t+1} is a random variable whose distribution is $Q(\cdot|s_t - h^*(s_t))$. Let $\mathscr{M}(S)$ be the set of probability measures on S endowed with the standard first order stochastic dominance. By δ_s denote a delta Dirac measure concentrated on s. Define $\Lambda : \mathscr{M}(S) \mapsto \mathscr{M}(S)$ by:

$$\Lambda(\mu)(\cdot) := \int_S Q(\cdot|s - h^*(s))\mu(ds).$$

Corollary 1 (Monotone transition dynamics)**.** *Let $h^* \in CM$ be a given SMPE. Then:*

1. *The set of invariant distributions of the process $s_{t+1} \sim Q(\cdot|s_t - h^*(s_t))$ is a complete lattice.*

2. *The sequence of $(\mu_n)_{n=1}^{\infty}$, where $\mu_{n+1} = \Lambda(\mu_n)$ with $\mu_0 = \delta_{\bar{S}}$, is convergent to the greatest invariant distribution of Λ, say $\bar{\mu}$.*

3. *Moreover, for any $s_0 \in S$ the sequence $(\mu_n)_{n=1}^{\infty}$ where $\mu_{n+1} = \Lambda(\mu_n)$ with $\mu_0 = \limsup_k \Lambda^k(\delta_{s_0})$ is convergent to the invariant distribution of Λ.*

Proof. Recall that $\mathscr{M}(S)$ is a complete lattice with a smallest element δ_0 and a greatest element $\delta_{\bar{S}}$ (see Kamae, Krengel, and O'Brien (1977)). By Theorem 1, Λ is increasing on $\mathscr{M}(S)$. By Tarski fixed point theorem, Λ has a complete lattice of fixed points. Λ is also weakly continuous. Hence, by Tarski-Kantorovitch theorem (see, *e.g.*, theorem 1.2 in Dugundji and Granas (1982)) $\Lambda^n(\delta_{\bar{S}})$ is convergent to the greatest invariant distribution $\bar{\mu}$. The last point follows from Olszewski (2021) theorem applied to monotone and continuous mapping Λ on a complete lattice $\mathscr{M}(S)$. □

Observe that the expected allocations associated with the invariant distribution, *e.g.*, $\bar{\mu}$ are $c_1^* = \int_S \beta(h^*(s'))\bar{\mu}(ds')$ and $c_2^* = \int_S \gamma(h^*(s'))\bar{\mu}(ds')$. Conditions for uniqueness of invariant distributions can also be developed using standard arguments (see e.g. Futia (1982)).

3 Generalized Euler equations and the first-order characterization

We now aim to characterize SMPE using (generalized) Euler equations. To proceed, first observe that Theorem 1 proves existence of SMPE in CM, a set of Lipschitz continuous functions with Lipschitz constant 1. This implies that $h^* \in CM$ is a.e. differentiable. To proceed, we need, however, some further differentiability assumptions on $u^i(\cdot)$.

Assumption 3. *Each u^i is twice continuously differentiable, $(u^i)''(c) < 0$ for $c > 0$, and $\lim_{c \to 0}(u^i)'(c) = \infty$*

Recalling that $u_0(c) := \max_{c_1, c_2} u^1(c_1) + \eta u^2(c_2)$ s.t. $c_1 + c_2 = c$, we then obtain for $c > 0$:

$$(u^1)'(\beta(c)) = \eta(u^2)'(c - \beta(c)) = \eta(u^2)'(\gamma(c)).$$

We immediately have:

Lemma 8. *Assume 1 and 3. Then u_0 is twice continuously differentiable and β and γ are interior and continuously differentiable. As a result any u_t is also continuously differentiable.*

Its proof follows from Lemma 10. Indeed, let $c > 0$ and from the envelope theorem:

$$u_0'(c) = (u^1)'(\beta(c)) = \eta(u^2)'(\gamma(c)).$$

Hence and by the Implicit Function Theorem both β' and γ' exist and

$$\gamma'(c) = \frac{(u^1)''(\beta(c))}{(u^1)''(\beta(c)) + \eta(u^2)''(\gamma(c))}.$$

Clearly $\beta'(c) + \gamma'(c) = 1$ and:

$$
\begin{aligned}
u_t'(c) &= (u^1)'(\beta(c))\beta'(c) + \eta \left(\frac{\delta_2}{\delta_1}\right)^t (u^2)'(\gamma(c))\gamma'(c) \\
&= u_0'(c) - \eta \left[1 - \left(\frac{\delta_2}{\delta_1}\right)^t\right] (u^2)'(\gamma(c))\gamma'(c).
\end{aligned}
$$

The previous relation will be particular useful in our generalized Euler equation. It allows to control the effects of the current investment on utilities of the next generations. The time-consistency problem is also visible in the above equation. Indeed, unless $\eta = 0$ or $\delta_2 = \delta_1$, $u_t'(c) \neq u_0'(c)$, and the corrective factor needs to be applied in the Euler equation to account for changing preferences. Now define:

$$
\begin{aligned}
F_V(x) &:= \int_S V(s')Q(ds'|x), \\
F_V'(x) &:= \frac{d}{dx} \int_S V(s')Q(ds'|x).
\end{aligned}
$$

Moreover, note that since $U(h^*) \in \mathscr{V}$ and $F_{U(h^*)}$ is bounded and concave hence a.e. differentiable. To proceed on $F_V'(x)$, we need an extra assumption:

Assumption 4. *For any $V \in \mathscr{V}$, the function F_V is twice continuously differentiable on the interior of S and* $\lim_{i \to 0} F_V(i) = \infty$.

We illustrate the above assumption in the following example.

Example 3. Continue Example 2 and consider g that is twice continuously differentiable and satisfy the Inada condition. Then Assumption 4 is satisfied.

Recalling the definition of $F_V'(x)$, consider then the following first order condition for interior SMPE h^*:

$$
u_0'(h^*(s)) = \delta_1 F_{U(h^*)}'(s - h^*(s)). \tag{4}
$$

Clearly, the right hand side of the above condition involves the derivative of the continuation value $U(h^*)$. The principle of optimality being not applicable due to time-inconsistency, the differentiability of $U(h^*)$ requires a differentiable h^*. The latter just follows from Lemma 10, and thus Assumptions 1 and 2, with $f_1 = u_0$ and $f_2(\cdot) = \delta_1 F_{U(h^*)}(\cdot)$. This is asserted in the following statement, whose proof we omit.

Theorem 2. *Under Assumptions 1, 2, 3 and 4, there exists a SMPE h^* that is differentiable on the interior of S.*

In what follows we characterize further the expression $F'_{U(h^*)}(s-h^*(s))$. For this reason we need a (differentiable) density representation for Q.

Assumption 5. *There exists a measure $\rho : S \times S \mapsto \mathbb{R}$ such that*

(i) *For $i \in S \setminus \{0\}$, $Q(A|i) = \int_A \rho(i, s')ds'$.*

(ii) *For any $i \in S \setminus \{0\}$, $\rho(i, \cdot)$ is a density of a probability measure with respect to the standard measure, i.e. $\int_S \rho(i, s')ds' = 1$;*

(iii) *For Lebesgue a.e. $s \in S$, $R(i, s) = \int_0^s \rho(i, s')ds'$ is twice continuously differentiable at any $i > 0$ and $s \mapsto \sup_{i\in(0,s]} R'_1(i, s)$ is integrable where $R'_1(i, s) := \partial R(i, s)/\partial i$;*

In the following derivations we repeatedly use the following result on integration by parts.

Lemma 9. *Let $V \in \mathscr{V}$:*

(i) *V' exists almost everywhere and $V(s) = \int_0^s V'(s')ds'$ for any $s \in S$.*

(ii) $\displaystyle \frac{\partial}{\partial i}\int_S V(s)\rho(i, s)ds = -\int_S V'(s)R'_1(i, s)ds.$

Let h^* be a SMPE and let us also define:

$$
\begin{aligned}
v_t^*(s) &:= E_s \sum_{\tau=t}^{\infty} \delta_1^{\tau-t} u_\tau \left(h^*(s_\tau)\right), \\
I_t^*(s) &= -\int_S (v_t^*)'(s')R'_1(s-h^*(s), s')ds',
\end{aligned}
$$

whenever the derivative exists. Recall further that:

$$(5) \quad I_1^*(s) = F'_{U(h^*)}(s - h^*(s)).$$

The next theorem together with equations (4) and (5) constitute our first order characterization of a differentiable SMPE h^*.

Theorem 3. *Under Assumptions 1, 2, 3, 4 and 5, each I_t^* is well defined and obeys:*

$$(6) \quad I_t^*(s) = -\int_S u_t'(h^*(s'))(h^*)'(s')R_1'(s - h^*(s), s')ds' \\ - \delta_1 \int_S I_{t+1}^*(s')(1 - (h^*)'(s'))R_1'(s - h^*(s), s')ds'.$$

Proof. We show it is differentiable in s. Observe that v_t^* is the fixed point of the contraction mapping $T^{h^*} := (T_t^{h^*})_{t=1}^{\infty}$ such that $T^{h^*} : \mathscr{V}^\infty \mapsto \mathscr{V}^\infty$ and is defined as follows:

$$T_t^{h^*}(v)(s) = u_t(h^*(s)) + \delta_1 \int_S v_{t+1}(s')\rho(s - h^*(s), s')ds'.$$

By Theorem 2 it follows that h^* is differentiable and by Assumption 5 the second term on the right hand side is differentiable for any $v_{t+1} \in \mathscr{V}$. Hence v_t^* is differentiable for any t.

By Lemma 9 for any $i > 0$ we have

$$(7) \quad \frac{d}{di}\int_S v_{t+1}^*(s')\rho(i, s')ds' = -\int_S (v_{t+1}^*)'(s)R_1'(i, s')ds'.$$

Combining definition of v_t^*, T^{h^*}, equation (7) and Theorem 2

$$(v_t^*)'(s') = u_t'(h^*(s'))(h^*)'(s') - \delta_1(1 - (h^*)'(s'))\int_S (v_{t+1}^*)'(s'')R_1'(s' - h^*(s'), s'')ds'' \\ = u_t'(h^*(s'))(h^*)'(s') + \delta_1(1 - (h^*)'(s'))I_{t+1}^*(s').$$

Multiplying both sides above by $-R_1'(s - h^*(s), s')$ and integrating over ds', the equation (6) holds. □

Remark 2 (Relations to quasi-hyperbolic discounting). The generalized Euler equation characterization of differentiable h^* (as specified in equations (4), (5) and (6)) requires a comment with respect to related characterizations for SMPE in a class of quasi-hyperbolic discounting models. Harris and Laibson (2001) and more recently Balbus, Reffett, and Woźny (2018) obtained a FOC of the differentiable equilibrium investment policy in such models under strong assumptions on the stochastic transition. Relative to the first order characterization in quasi-hyperbolic discounting models, the one considered in the collective household problem is significantly more complicated. Indeed, equation (6) constructs an infinite sequence of the derivatives $(I_t^*)_t$ that are necessary to compute I_1^*, i.e. the right hand side of the FOC (4). This is in stark contrast to the quasi-hyperbolic discounting models where such construction is not necessary[10] due to the assumption that the preferences in the quasi-hyperbolic discounting model form the next period on are exponentially discounted by δ and hence stationary. This difference results from the inherently non-stationary structure of the collective household model, where further within period preferences $(u_t)_t$ are governed by (3).

To circumvent this non-stationarity complication, in section 4 we will provide a way to approximate sequences of $(I_t^*)_t$ using FOCs for equilibria in a constructed sequence of bequest games.

4 Bequest games, altruism and approximation

We now provide an algorithm to approximate SMPE by considering a sequence of the corresponding bequest games with paternalistic altruism restricted to finite (but growing) number of the consecutive generations.

For this reason consider an infinite horizon, bequest game with one period ahead paternalistic altruism. We refer the reader to the classic Phelps and Pollak (1968) paper.[11] That is, each generation t derives utility from own consumption c_t and *consumption* of the next generation c_{t+1} and evaluates both using the following preferences: $u_0(c_t)+\delta_1 u_1(c_{t+1})$. Assume first that each u_t is defined as in equation (3). Each period generation inherits

[10]See e.g. page 307 in Balbus, Reffett, and Woźny (2018) for a direct comparison.

[11]But see also Leininger (1986), Amir (1996), Nowak (2006), Balbus, Reffett, and Woźny (2013) or Balbus, Jaśkiewicz, and Nowak (2015) for some related contributions.

output / state s_t and a stochastic state transition is governed as before by $s_{t+1} \sim Q(\cdot|s_t - c_t)$. We assume 1 and 2 and are now seeking for a SMPE $h_1^* \in CM$ such that for any $s \in S$ we have:

$$h_1^*(s) \in \arg\max_{c\in[0,s]} \left\{ u_0(c) + \delta_1 \int_S u_1(h_1^*(s'))Q(ds'|s-c) \right\}.$$

Under Assumptions 1 and 2 and by Theorem 1, this existence result is available.

Next, we consider a similar game but with two period ahead paternalistic altruism, i.e. where each generation derives utility from own consumption c_t and consumption of the two consecutive generations c_{t+1}, c_{t+2} and evaluates them according to: $u_0(c_t) + \delta_1 u_1(c_{t+1}) + \delta_1^2 u_2(c_{t+2})$. As before, assuming 1 and 2 we can show existence of SMPE $h_2^* \in CM$, *i.e.*,

$$h_2^*(s) \in \arg\max_{c\in[0,s]} \left\{ u_0(c) + \delta_1 \int_S \left[u_1(h_2^*(s')) + \delta_1 \int_S u_2(h_2^*(s''))Q(ds''|s' - h_2^*(s')) \right] Q(ds'|s-c) \right\}.$$

Next, for any $h \in CM$ and $n \in \mathbb{N}$, let

$$U_n(h)(s) := E_s \left(\sum_{t=1}^{n} (\delta_1)^{t-1} u_t(h(s_t)) \right).$$

Continuing as above we can consider a sequence of bequest games with n-period ahead paternalistic altruism and for each consider a SMPE $h_n^* \in CM$. That is

$$h_n^*(s) \in \arg\max_{c\in[0,s]} \left\{ u_0(c) + \delta_1 \int_S U_n(h_n^*)(s')Q(ds'|s-c) \right\}.$$

Its existence is guaranteed by the arguments similar to the one used in the construction of Theorem 1. We conclude with the following result:

Theorem 4. *For any $n \in \mathbb{N}$ there exists $h_n^* \in CM$, a SMPE in the model with n period ahead paternalistic altruism. The sequence $(h_n^*)_n$ has a converging subsequence, whose limit $h^* \in CM$ is a SMPE of the collective household problem (according to Definition 1).*

This is our main approximation result. It shows how a sequence of equilibria in such a family of the bequest games can be used to approximate the equilibrium of the collective household model as analyzed in this paper. We now present its proof and later discuss its implications on the generalized Euler equation characterization.

Proof. The proof of existence is similar to the one of Theorem 1, the only difference being that we apply Lemma 11 that establishes the continuity of the operator $U_n : CM \mapsto \mathscr{V}$. So the equilibrium $h_n^* \in CM$ exists for any n. Observe that:

$$(8) \quad U_n(h)(s) := \sum_{t=1}^{n} (\delta_1)^{t-1} E_s(u_t(h(s_t))) = \sum_{t=1}^{n} (\delta_1)^{t-1} \int_S u_t(h(s'))Q_h^t(ds'|s)$$

for every $h \in CM$ and $n \in \mathbb{N}$. Here Q_h^t is the standard $t-$ step transition probability generated by h, that is $Q_h^1(\cdot|s) := Q(\cdot|s - h(s))$ and for $t \geq 1$ we apply by the standard Chapman-Kolmogorov equations (see chapter 4.2 in Ross (1997)):

$$(9) \quad Q_h^{t+1}(\cdot|s - h(s)) = \int_S Q_h^t(\cdot|s')Q(ds'|s - h(s)).$$

We claim that if $h_n \rightrightarrows h$ and $s_n \to s$, then $Q_{h_n}^t(\cdot|s_n) \to Q_h^t(\cdot|s)$ weakly for every t. For $t = 1$, it follows directly by Assumption 2. Suppose that this thesis holds for some t. Then for any continuous and bounded function f and any sequence s'_n such that $s'_n \to s'$ as $n \to \infty$ we have

$$\int_S f(s'')Q_{h_n}^t(ds''|s'_n) \to \int_S f(s'')Q_h^t(ds''|s').$$

As a result, by Assumption 2 we have

$$\int_S f(s'')Q_{h_n}^t(ds''|\cdot) \rightrightarrows \int_S f(s'')Q_h^t(ds''|\cdot).$$

Hence, by induction hypothesis, (9) (and Corollary 15.7 in Aliprantis and Border (2006)) the thesis holds for Q^{t+1}, hence for all t. As a result, since by Assumption 1 and Lemma 10, $u_t(h_n(\cdot)) \rightrightarrows u_t(h(\cdot))$, hence, by (8) and by the Dominated Convergence Theorem

$U_n(h_n)(s_n) \to U(h)(s)$ as $n \to \infty$. Now, passing to subsequences if necessary suppose $h_n^* \rightrightarrows h^*$. Hence

$$(10)\quad \lim_{n\to\infty} \left(u_0(h_n^*(s)) + \delta_1 \int_S U_n(h_n^*)(s')Q(ds'|s - h_n^*(s)) \right) = u_0(h^*(s)) + \delta_1 \int_S U(h^*)(s')Q(ds'|s - h^*(s)).$$

Observe that for all $n \in \mathbb{N}$ and $c \in [0, s]$ we have

$$(11)\quad u_0(h_n^*(s)) + \delta_1 \int_S U_n(h_n^*)(s')Q(ds'|s - h_n^*(s)) \geq u_0(c) + \delta_1 \int_S U_n(h_n^*)(s')Q(ds'|s - c).$$

Taking the limit in (11), by (10) we conclude that

$$u_0(h^*(s)) + \delta_1 \int_S U(h^*)(s')Q(ds'|s - h^*(s)) \geq u_0(c) + \delta_1 \int_S U(h^*)(s')Q(ds'|s - c),$$

for any $c \in [0, s]$. □

Observe that we can also use the generalized FOCs for such an approximation. Indeed under Assumptions 3, 4 and 5, each $h_n^* \in CM$, *i.e.*, each SMPE of the bequest game with n period ahead altruism, can be characterized by the following FOC:

$$u_0'(h_n^*) = \delta_1 F'_{w_n^*}(s - h_n^*(s)),$$

where $w_n^*(s) := E_s \sum_{\tau=1}^{n} (\delta_1)^\tau u_\tau(h_n^*(s_\tau))$ and $F'_{w_n^*}(s - h_n^*(s))$ can be computed using a finite sequence of $(I_t^*)_{t=1}^n$ using equation (6) and taking $I_{n+1}^* \equiv 0$. This proposes a way to approximate the FOCs of the collective household equilibrium using a sequence of FOCs of the approximating sequence of bequest games.

A Appendix

A.1 Auxiliary results

We start by continuing example 1 and presenting the graph depicting lack of the argument maximizing f on $[0, 4]$ for $h(s) = s$ for all $s \le 1$ and $h(s) = s/2$ for $s > 1$ and $s = 4$.

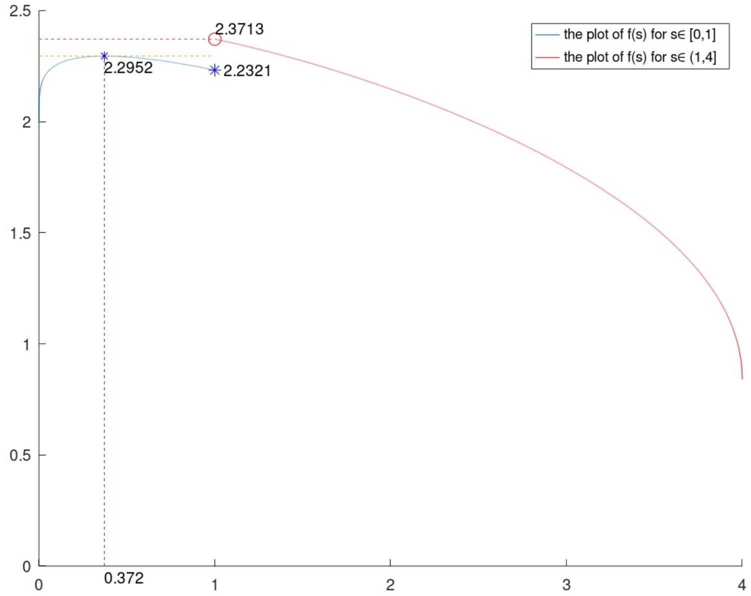

Figure 1: Payoff f from Example 1 as a function of $i \in [0, 4]$.

We first establish a general purpose technical lemma that is repeatedly used in our paper.

Lemma 10. *Let f_1 and f_2 be increasing, concave functions on $[0, \infty)$ and let at least one of them be strictly concave. Let $a(s), b(s)$ be such that*

$$f^*(s) = f_1(a(s)) + f_2(b(s)) = \max\left\{f_1(c_1) + f_2(c_2) : c_1 \ge 0, c_2 \ge 0, c_1 + c_2 = s\right\}.$$

Then

(i) *Both a and b are uniquely determined, increasing and Lipschitz continuous functions (with a constant 1);*

(ii) *f^* is strictly concave and continuous whenever both f_1 and f_2 are strictly concave;*

(iii) *If f_1 and f_2 are twice continuously differentiable on $(0,\infty)$, $\lim_{s\downarrow 0} f_1'(s) = \lim_{s\downarrow 0} f_2'(s) = \infty$, and $f_1''(s) < 0$ for $s > 0$, then a and b are differentiable at any $s > 0$ and $a(s) \in (0,s)$, the same for $b(s)$. As a result f^* is twice continuously differentiable on $(0,\infty)$.*

Proof. Proof of (i). Since f_1 and f_2 are increasing, concave and at least one of them is strictly concave both a and b are uniquely determined. Next we prove monotonicity. On the contrary suppose that a is not increasing, i.e. there is are $0 \le s_1 < s_2$ such that $a_1 := a(s_1) > a(s_2) := a_2 \ge 0$. Put $\Delta := a_1 - a_2$. Then $\Delta > 0$. We have then

$$\begin{aligned} 0 &\le f_1(a_1-\Delta) - f_1(a_1) + f_2(\Delta + s_2 - a_1) - f_2(s_2 - a_1) \\ &< f_1(a_1-\Delta) - f_1(a_1) + f_2(\Delta + s_1 - a_1) - f_2(s_1 - a_1) \\ &= f_1(a_2) + f_2(s_1 - a_2) - f_1(a_1) - f_2(s_1 - a_1) \le 0. \end{aligned}$$

This is a contradiction, as a is optimal and hence $f_1(a_2)+f_2(s_1-a_2) \le f_1(a_1)+f_2(s_1-a_1)$. As a result we obtain that a is increasing. Similarly we prove b is increasing. The rest of the proof that both a and b are Lipschitz with 1 is standard.

Proof of (ii). We show f^* is strictly increasing. Let us continue notations for s_1, s_2, a_1, and a_2 from (i). If $a_1 = a_2$, then the thesis is done. By (i) we may assume $a_1 < a_2$. Then the function

$$f^*(s_1) < f_1(a_1) + f_2(s_2 - a_1) \le f^*(s_2).$$

Hence f^* is strictly increasing. Now we show the strict concavity: Let $\alpha \in (0,1)$, $s^\alpha = \alpha s_1 + (1-\alpha)s_2$, and $a^\alpha = \alpha a_1 + (1-\alpha)a_2$. By strict concavity of f_1 and f_2 we have

$$\begin{aligned} f^*(s^\alpha) &\ge f_1\left(a^\alpha\right) + f_2\left(s^\alpha - a^\alpha\right) \\ &> \alpha\left(f_1\left(a_1\right) + f_2\left(s_1 - a_1\right)\right) + (1-\alpha)\left(f_1\left(a_2\right) + f_2\left(s_2 - a_2\right)\right) \\ &= \alpha f^*(s_1) + (1-\alpha) f^*(s_2), \end{aligned}$$

and hence f^* is strictly concave.

Proof of (iii). It is routine to verify $a(s) \in (0, s)$ and the same for b for any $s > 0$ and the function $a(s)$ solves the equation

$$(12) \quad f_1'(c) - f_2'(s-c) = 0$$

with respect to c. Observe that

$$(13) \quad f_1''(c) + f_2''(s-c) < 0$$

for any $c \in (0, s)$. Observe both f_1'' and f_2'' are not positive and $f_1''(c) < 0$ for any $c \in (0, s)$. hence the strict inequality in (13) holds for a neighborhood of $c = a(s)$. By Implicit Function Theorem and (12) we obtain the differentiability of a at any $s > 0$. Consequently b is differentiable and by Envelope Theorem $(f^*)'(s) = f_1'(a(s))$ holds, hence f^* has the second derivative

$$(f^*)''(s) = f_1''(a(s))a'(s).$$

□

A.2 Proofs

Proof of Lemma 3. Let $v \in \mathscr{V}^\infty$. It is easy to see that by Assumptions 1 and 2, $T_t^h(v)$ is a continuous function, and $T^h : \mathscr{V}^\infty \mapsto \mathscr{V}^\infty$. We show that, T^h is a contraction with respect to the metric induced by the norm $\|\cdot\|^\kappa$. Let $v^1, v^2 \in \mathscr{V}^\infty$.

$$\begin{aligned}
\|T^h(v^1) - T^h(v^2)\|^\kappa &= \sum_{t=1}^{\infty} \frac{\|T_t^h(v_{t+1}^1) - T_t^h(v_{t+1}^2)\|_\infty}{\kappa^{t-1}} \\
&\le \delta_1 \sum_{t=1}^{\infty} \frac{\|v_{t+1}^1 - v_{t+1}^2\|_\infty}{\kappa^{t-1}} \\
&\le \delta_1 \kappa \sum_{t=1}^{\infty} \frac{\|v_t^1 - v_t^2\|_\infty}{\kappa^{t-1}} \\
&= \delta_1 \kappa \|v^1 - v^2\|^\kappa.
\end{aligned}$$

□

Proof of Lemma 4. Let $v, v' \in \mathscr{V}^\infty$ and define

$$\phi_t(\cdot) := \int_S v_t(s') Q(ds'|\cdot).$$

By Assumption 2 it follows that ϕ_t is concave and continuous. Hence

$$\begin{aligned}
T_t^{h'}(v')(s) - T_t^h(v)(s) = u_t(h'(s)) - u_t(h(s)) + \delta_1 \int_S (v'_{t+1}(s') - v_{t+1}(s')) Q(ds'|s - h'(s)) \\
+ \delta_1(\phi_{t+1}(s - h'(s)) - \phi_{t+1}(s - h(s))).
\end{aligned}$$

Hence

$$\text{(14)} \quad \begin{aligned}
\|T^{h'}(v')(s) - T^h(v)\|^\kappa \le \sum_{t=1}^{\infty} \frac{|u_t(h'(s)) - u_t(h(s))|}{\kappa^{t-1}} \\
+ \delta_1 \kappa \left(\|v' - v\|^\kappa + \sum_{t=1}^{\infty} \frac{\phi_t(s - h'(s)) - \phi_t(s - h(s))}{\kappa^{t-1}} \right).
\end{aligned}$$

Since u_t and ϕ_t are concave and vanishing at 0, hence are subadditive. Hence, by (14) we have

$$\text{(15)} \quad \|T^{h'}(v')(s) - T^h(v)\|^\kappa \le \sum_{t=1}^{\infty} \frac{\psi_t(\|h' - h\|_\infty)}{\kappa^{t-1}} + \delta_1 \kappa \|v' - v\|^\kappa,$$

where $\psi_t(i) = u_t(i) + \delta_1 \kappa \phi_t(i)$ for $i \in S$. We have

$$\|\psi\|_\infty \leq (u^1(\bar{S}) + \eta u^2(\bar{S})) \left(1 + \frac{\kappa \delta_1}{1 - \delta_1}\right),$$

and any ψ_t is continuous at 0. Hence and by Weierstrass criterion for uniform convergence (Theorem 7.10 in Rudin (1964)), if $h' \rightrightarrows h$ then

$$\sum_{t=1}^{\infty} \frac{\psi_t(\|h' - h\|)}{\kappa^{t-1}} \to 0.$$

Hence and by (15), $\|T^{h'}(v')(s) - T^h(v)\|^\kappa \to 0$ as $\|v' - v\|^\kappa \to 0$ and $\|h' - h\|_\infty \to 0$. Hence $T^h(v)$ is continuous as a function of h and v. $\square$

Proof of Lemma 5. Observe that for $h, h' \in CM$ we have

$$\begin{aligned} ||v^{h'} - v^h||^\kappa &= ||T^{h'}(v^{h'}) - T^h(v^h)||^\kappa \\ &\leq ||T^{h'}(v^{h'}) - T^{h'}(v^h)||^\kappa + ||T^{h'}(v^h) - T^h(v^h)||^\kappa \\ &\leq \delta_1 \kappa ||v^{h'} - v^{h'}||^\kappa + ||T^{h'}(v^h) - T^h(v^h)||^\kappa, \end{aligned}$$

where the last inequality follows from Lemma 3 that any of $T^{h'}$ is a contraction mapping $\delta_1 \kappa$. Hence

$$||v^{h'} - v^h||^\kappa \leq \frac{1}{1 - \delta_1 \kappa} ||T^{h'}(v^h) - T^h(v^h)||^\kappa.$$

To finish the proof, using Lemma 4 we only need to take the limit $h' \rightrightarrows h$ above. Then, for any t, $v_t^{h'} \to v_t^h$. Since $v_1^{h'} = U(h')$ and $v_1^h = U(h)$, hence $U(h') \rightrightarrows U(h)$. $\square$

Proof of Lemma 7. By Lemma 3 it follows that for $h \in CM$, $U(h) \in \mathscr{V}$. Now apply Lemma 10 by setting u_0 for f_1, and $\delta_1 \int_S U(h)(s') Q(ds'|i)$ for f_2. Then clearly $a(s)$ from Lemma 10 is $BR(h)(s)$, and $b(s)$ is $s \mapsto s - BR(h)(s)$. Hence $BR(h)(s)$ and $s - BR(h)(s)$ are both increasing, and $BR(h) \in CM$. $\square$

Proof of Theorem 1. By Lemma 6 CM is compact and clearly convex subset of the set of bounded continuous functions. By Lemma 7, we conclude that BR maps CM into itself. We show that BR is a continuous operator on CM. Equivalently we show CM has a continuous graph. Fix s and put

$$\Pi(c,h) := u_0(c) + \delta_1 \int_S U(h)(s')Q(ds'|s-c).$$

Let $h_n \rightrightarrows h$ and assume $BR(h_n)(s) \to y$ as $n \to \infty$. By Lemma 5, $U(h_n) \rightrightarrows U(h)$, and by Feller property $Q(\cdot|s-h_n(s)) \to Q(\cdot|s-h(s))$ weakly as $n \to \infty$. Hence we have

$$(16) \quad \Pi(BR(h_n)(s), h_n) \to \Pi(y,h) \quad \text{and } \Pi(c,h_n) \to \Pi(c,h)$$

for arbitrary $c \in [0,s]$. By definition of $BR(h_n)$ we have

$$(17) \quad \Pi(c,h_n) \leq \Pi(BR(h_n)(s), h_n).$$

Combining (16) and (17)

$$\Pi(c,h) \leq \lim_{n\to\infty} \Pi(c,h_n) \leq \lim_{n\to\infty} \Pi(BR(h_n), h_n) = \Pi(y,h).$$

Since $\Pi(\cdot,h)$ has a unique maximum, hence $y = BR(h)$. As a result, BR is continuous on CM, hence by Schauder-Tychonof Theorem, there is at least one fixed point $h^* = BR(h^*)$. □

Proof of Lemma 9. Integrating by parts (Theorem 18.19 in Hewitt and Stromberg (1965))

$$\begin{aligned} \int_S V(s)\rho(i,s)ds &= V(\bar{S})R(i,\bar{S}) - V(0)R(i,0) - \int_S V'(s)R(i,s)ds \\ &= V(\bar{S}) - \int_S V'(s)R(i,s)ds. \end{aligned}$$

By Assumption 5 we can differentiate the equation above with respect to i and then obtain the thesis. □

The next lemma is used in the proof of our approximation theorem.

Lemma 11. *Assume 1 and 2. For any $n \in \mathbb{N}$, the operator $U_n : CM \mapsto \mathscr{V}$ is continuous.*

Proof. Observe that for $h \in CM$, $U_n(h) = (T^h)^n(\mathbf{0})$ (the n-th composition of 0). For $n = 1$ the thesis is obvious. Suppose that it is true for n and we only need to check the thesis for $n+1$. Let $h, h' \in CM$

$$\begin{aligned} ||U_{n+1}(h') - U_{n+1}(h)||^\kappa &= ||T^{h'}(U_n(h')) - T^h(U_n(h))||^\kappa \\ &\leq ||T^{h'}(U_n(h')) - T^{h'}(U_n(h))||^\kappa + ||T^{h'}(U_n(h)) - T^h(U_n(h))||^\kappa \\ &\leq \delta_1 ||U_n(h') - U_n(h)||^\kappa + ||T^{h'}(U_n(h)) - T^h(U_n(h))||^\kappa, \end{aligned} \tag{18}$$

where the last inequality follows from Lemma 3 that any of $T^{h'}$ is a contraction mapping $\delta_1\kappa$. Now we take a limit $h' \rightrightarrows h$. By induction hypothesis it follows that $||U_n(h') - U_n(h)||^\kappa \to 0$. By Lemma 3 it follows that $||T^{h'}(U_n(h)) - T^h(U_n(h))||^\kappa \to 0$. As a result, the left hand side in (18) tends to 0. □

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