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First year, first semester
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One dimensional analysis

1 Basic properties of the set of real numbers

$\mathbb{N}$ is the set of non-negative integers.

$\mathbb{Z}$ is the set of integers.

$\mathbb{Q}$ is the set of rational numbers p/q with p and q in $\mathbb{Z}$ and $q \neq 0$.

$\mathbb{R}$ is the field of real numbers with the basic operations, addition $+$ and multiplication and the complete order $\leq$.

For all $(x, y) \in \mathbb{R} \times \mathbb{R}$, with $x \leq y$,

- $[x, y] = \{z \in \mathbb{R} \mid x \leq z \leq y\}$,
- $[x, y[= \{z \in \mathbb{R} \mid x \leq z < y\}$,
- $]x, y] = \{z \in \mathbb{R} \mid x < z \leq y\}$,
- $]x, y[= \{z \in \mathbb{R} \mid x < z < y\}$,
- $[x, +\infty[= \{z \in \mathbb{R} \mid x \leq z\}$,
- $]x, +\infty[= \{z \in \mathbb{R} \mid x < z\}$,
- $]-\infty, y] = \{z \in \mathbb{R} \mid z \leq y\}$,
- $]-\infty, y[= \{z \in \mathbb{R} \mid z < y\}$.

We extend the order on $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ as follow: for all real numbers x, $-\infty < x < +\infty$.

For all $x \in \mathbb{R}$, $|x| = \max\{x, -x\}$ is the absolute value of x.

The distance between two elements x and y of $\mathbb{R}$ is $d(x, y) = |x - y|$. We remark that

- $d(x, y) = d(y, x)$,
- $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$ and,
- for all $(x, y, z) \in \mathbb{R}^3$, $d(x, y) \leq d(x, z) + d(z, y)$.

We also recall the useful inequality when we want to deal with the product:

$$|ab - cd| \leq |a||b - d| + |d||a - c|$$

A subset I of $\mathbb{R}$ is an interval if for all $(x, y) \in I \times I$, $[x, y] \subset I$.

For all $x \in \mathbb{R}$, $\{z \in \mathbb{R} \mid d(x, z) \leq r\} = [x - r, x + r]$ and $\{z \in \mathbb{R} \mid d(x, z) < r\} =]x - r, x + r[$.

For all $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $x \leq y$, $[x, y] = \{z \in \mathbb{R} \mid d(z, \frac{x+y}{2}) \leq \frac{y-x}{2}\}$ and $]x, y[= \{z \in \mathbb{R} \mid d(z, \frac{x+y}{2}) < \frac{y-x}{2}\}$.

A subset A of $\mathbb{R}$ is bounded if there exists $r \geq 0$ such that $A \subset [-r, r]$.

x is an upper bound of A if $A \subset]-\infty, x]$ or equivalently if for all $a \in A$, $a \leq x$.

x is a lower bound of A if $A \subset [x, +\infty[$ or equivalently if for all $a \in A$, $a \geq x$.

The supremum of A is the least upper bound of A. The infimum of A is the greatest lower bound of A. The supremum of A is the maximum of A if it belongs to A. The infimum of A is the minimum of A if it belongs to A.

Fundamental property of $\mathbb{R}$. The key property of the set $\mathbb{R}$ is the following: all nonempty bounded above subsets of $\mathbb{R}$ have a supremum.

Furthermore, $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that for all $a \in \mathbb{R}$ and for all $r > 0$, there exists $b \in \mathbb{Q} \cap]a - r, a + r[$. In some sense, which can be precisely defined, $\mathbb{R}$ is the smallest set containing $\mathbb{Q}$ which satisfies the existence of a supremum for all bounded above nonempty subsets.

Exercise 1 Show that all bounded below subsets of $\mathbb{R}$ has an infimum. Hint: if A is a bounded below subset of $\mathbb{R}$, consider the set $-A = \{-a \mid a \in A\}$, show that it is bounded above and show that the opposite of the supremum of $-A$ is an infimum of A.

2 Sequences

Definition 1 A sequence is a mapping from $\mathbb{N}$ to $\mathbb{R}$.

A sequence is often denoted (u_n) where u_n is the image of $n \in \mathbb{N}$.
Examples

Definition 2 A sequence (u_n) is

a) increasing if for all $n \in \mathbb{N}$, $u_{n+1} \geq u_n$;

b) decreasing if for all $n \in \mathbb{N}$, $u_{n+1} \leq u_n$;

c) strictly increasing if for all $n \in \mathbb{N}$, $u_{n+1} > u_n$;

d) strictly decreasing if for all $n \in \mathbb{N}$, $u_{n+1} < u_n$;

e) bounded if there exists $r > 0$ such that for all $n \in \mathbb{N}$, $u_n \in [-r, r]$.

Definition 3 Let (u_n) and (v_n) be two sequences and $t \in \mathbb{R}$.

a) The sequence (w_n) defined by for all $n \in \mathbb{N}$, $w_n = u_n + v_n$ is called the sum of (u_n) and (v_n).

b) The sequence (w_n) defined by for all $n \in \mathbb{N}$, $w_n = u_n v_n$ is called the product of (u_n) and (v_n).

c) The sequence (w_n) defined by for all $n \in \mathbb{N}$, $w_n = t u_n$ is called the product of t and (u_n).

d) If for all $n \in \mathbb{N}$, $u_n \neq 0$, then we can define a sequence (w_n) by $w_n = 1/u_n$ for all $n \in \mathbb{N}$.

Definition 4 The sequence (u_n) converges to a limit $\ell \in \mathbb{R}$ if for all $r > 0$, there exists an integer $n_r \in \mathbb{N}$ such that for all $n \geq n_r$, $u_n \in]\ell - r, \ell + r[$.

If a sequence converges to a limit, we say that it is convergent and the limit is denoted by $\lim_{n\to\infty} u_n$.

Proposition 1 *(i) If a sequence is convergent, it has unique limit.*

(ii) The sequence (u_n) converges to the limit ℓ if and only if the sequence $(|u_n - \ell|)$ converges to 0.

(iii) Let (u_n) be a convergent sequence and $a \in \mathbb{R}$. If for all $n \in \mathbb{N}$, $u_n \leq a$, then $\lim_{n\to\infty} u_n \leq a$. If for all $n \in \mathbb{N}$, $u_n \geq a$, then $\lim_{n\to\infty} u_n \geq a$.

(iv) If the sequence (u_n) is convergent, then it is bounded.

Exercise 2 Let (u_n) and (v_n) be two sequences. We assume that (u_n) is convergent. Show that if the set $\{n \in \mathbb{N} \mid u_n \neq v_n\}$ is finite, then, (v_n) is convergent and has the same limit than (u_n).

We assume that (u_n) is not convergent. Show that if the set $\{n \in \mathbb{N} \mid u_n \neq v_n\}$ is finite, then, (v_n) is not convergent.

Proposition 2 *Let (u_n) and (v_n) be two sequences and $t \in \mathbb{R}$. We assume that (u_n) converges to ℓ and (v_n) converges to ℓ'. Then*

a) The sequence $(u_n + v_n)$ converges to $\ell + \ell'$.

b) The sequence $(u_n v_n)$ converges to $\ell\ell'$.

c) The sequence (tu_n) converges to $t\ell$.

d) The sequence $(|u_n|)$ converges to $|\ell|$.

e) The sequence $(\max\{u_n, v_n\})$ converges to $\max\{\ell, \ell'\}$.

f) The sequence $(\min\{u_n, v_n\})$ converges to $\min\{\ell, \ell'\}$.

g) If for all $n \in \mathbb{N}$, $u_n \neq 0$ and $\ell \neq 0$, then the sequence $(\frac{1}{u_n})$ converges to $\frac{1}{\ell}$.

Definition 5 The sequence (u_n) converges to $+\infty$ if for all $r \in \mathbb{R}$, there exists an integer $n_r \in \mathbb{N}$ such that for all $n \geq n_r$, $u_n \in]r, +\infty[$.

The sequence (u_n) converges to $-\infty$ if for all $r \in \mathbb{R}$, there exists an integer $n_r \in \mathbb{N}$ such that for all $n \geq n_r$, $u_n \in]-\infty, r[$.

Calculus rules with infinite limits

$\lim_{n\to\infty}(u_n + v_n)$

$\lim_{n\to\infty} u_n$ / $\lim_{n\to\infty} v_n$	$-\infty$	ℓ	$+\infty$
$-\infty$	$-\infty$	$-\infty$	?
ℓ'	$-\infty$	$\ell + \ell'$	$+\infty$
$+\infty$	?	$+\infty$	$+\infty$

$\lim_{n\to\infty}(u_n v_n)$

$\lim_{n\to\infty} u_n$ / $\lim_{n\to\infty} v_n$	$-\infty$	$\ell < 0$	0	$\ell > 0$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	?	$-\infty$	$-\infty$
$\ell' < 0$	$+\infty$	$\ell\ell'$	0	$\ell\ell'$	$-\infty$
0	?	0	0	0	?
$\ell' > 0$	$-\infty$	$\ell\ell'$	0	$\ell\ell'$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	?	$+\infty$	$+\infty$

Example $u_{n+1} = au_n + b$

Particular cases $b = 0$, $a = 1$, general case, geometric sequence, formula, arithmetic sequence.

Proposition 3 *Let (u_n) and (v_n) be two sequences.*

(i) If (u_n) is increasing and bounded above, then it is convergent.

(ii) If (u_n) is decreasing and bounded below, then it is convergent.

(iii) If (u_n) is increasing, (v_n) is decreasing and $(v_n - u_n)$ converges to 0, then (u_n) and (v_n) are convergents and they have the same limit. In that case, we say that the two sequences are adjacent.

Exercise 3 Let (u_n) be a bounded sequence. Let (v_n) be defined by, for all $n \in \mathbb{N}$, $v_n = \sup\{u_k \mid k \geq n\}$ and (w_n) be defined by, for all $n \in \mathbb{N}$, $w_n = \inf\{u_k \mid k \geq n\}$.

Show that (v_n) is decreasing, (w_n) is increasing and, for all $n \in \mathbb{N}$, $w_n \leq u_n$. Show that (v_n) and (w_n) are convergent.

Let us assume that (u_n) is decreasing. Show that, for all $n \in \mathbb{N}$, $v_n = u_n$ and $w_n = \lim_{n\to\infty} u_n$.

We are now back to the general case. Show that if (u_n) is convergent, then (v_n) and (w_n) converge to $\lim_{k\to\infty} u_k$.

Show that if $\lim_{n\to\infty} w_n < \lim_{n\to\infty} v_n$, then (u_n) is not convergent.

Cauchy Criterion: From the above exercise, we can derive the following very important criterion for convergence.

Proposition 4 *A sequence (u_n) is convergent if and only if it satisfies the following Cauchy criterion:*

$$\forall r > 0, \exists n \in \mathbb{N}, \forall p, q \geq n, |u_p - u_q| \leq r$$

From a given sequence (u_n), we can build many others by picking only some terms of it.

Definition 6 Let (u_n) be a real sequence. A subsequence of (u_n) is a sequence (v_n) defined by a strictly increasing mapping φ from $\mathbb{N}$ to itself and for all $n \in \mathbb{N}$, $v_n = u_{\varphi(n)}$.

Proposition 5 *If (u_n) is a converging sequence, then all subsequences of (u_n) are convergent and they are converging to the same limit.*

Exercise 4 Let (u_n) be a bounded sequence. Let (v_n) be defined by, for all $n \in \mathbb{N}$, $v_n = \sup\{u_k \mid k \geq n\}$. Let ℓ be the limit of (v_n). (We have prove in Exercise 3 that this sequence is convergent.) Show that there exists a subsequence (w_n) of (u_n), which converges to ℓ.

From Exercise 4, we deduce the fundamental Bolzano-Weierstrass Theorem

Theorem 1 *All bounded real sequences have a converging subsequence.*

From this result, we deduce a new convergence criterion for bounded sequences.

Proposition 6 *Let (u_n) be a bounded sequence. (u_n) is convergent if and only if all convergent subsequences of (u_n) have the same limit.*

Definition 7 Let (u_n) be a real sequence. $c \in \mathbb{R}$ is a cluster point of (u_n) if for all $r > 0$, the set $\{n \in \mathbb{N} \mid u_n \in]c - r, c + r[\}$ is infinite.

Proposition 7 *Let (u_n) be a real sequence. $c \in \mathbb{R}$ is a cluster point of (u_n) if and only if there exists a convergent subsequence (v_n) of (u_n) such that c is the limit of (v_n).*

Note that the Bolzano-Weierstrass Theorem can be equivalently stated as all bounded real sequences have a cluster point. We also deduce from the previous results that a bounded sequence is convergent if and only if it has a unique cluster point. In that case, the limit is the unique cluster point.

Example of recursive sequences of order 1

$u_{n+1} = f(u_n)$, convergence of the Newton algorithm.

3 Series

Let (u_n) be a real sequence. The series associated to (u_n) is the sequence (σ_n) defined by $\sigma_n = \sum_{\nu=0}^{n} u_\nu$.

Definition 8 The series associated to (u_n) (or, in short, the series (u_n)) is convergent if the sequence (σ_n) defined by $\sigma_n = \sum_{\nu=0}^{n} u_\nu$ is convergent.

The series associated to (u_n) is absolutely convergent if the sequence $(\sum_{\nu=0}^{n} |u_\nu|)$ is convergent.

Remark 1 One easily shows (exercise) that if the series (u_n) is convergent, then the sequence (u_n) converges to 0. The converse is not true. For example, show that the series associated to the sequence $(u_n = \frac{1}{n+1})$ is not convergent. Hint: show that the sequence $\sum_{\nu=0}^{n} \frac{1}{\nu+1}$ does not satisfy the Cauchy Criterion.

Using the Cauchy criterion of convergence, one has the fundamental following result.

Proposition 8 *If the series associated to (u_n) is absolutely convergent, then the series associated to (u_n) is convergent.*

Since the series associated to a non-negative sequence is increasing, we get the simple convergence criteria.

Proposition 9 *The series associated to the sequence (u_n) is absolutely convergent if and only if the sequence $(\sum_{\nu=0}^{n} |u_\nu|)$ is bounded above.*

Exercise 5 Let (u_n) and (v_n) be two sequences such that the associated series are convergent. Show that the series associated to $(u_n + v_n)$ is also convergent and that its limit is the sum of the limits of the series associated to (u_n) and (v_n).

Exercise 6 Show that the series associated to the sequence $((-1)^n \frac{1}{n+1})$ is convergent but not absolutely convergent.

Show that the series associated to the sequence (k^n) is absolutely convergent when $k \in\ ]-1, 1[$.

Proposition 10 *Let (u_n) be a decreasing sequence such that $\lim_{n\to\infty} u_n = 0$ and for all $n \in \mathbb{N}$, $u_n u_{n+1} \leq 0$. Then the series associated to the sequence (u_n) is convergent.*

Proposition 11 *Let (u_n) and (v_n) be two sequences with non-negative terms. We assume that the series associated to (v_n) is convergent.*

a) If for all $n \in \mathbb{N}$, $v_n > 0$ and $(\frac{u_n}{v_n})$ is bounded above then the series associated to (u_n) is convergent.

b) If for all $n \in \mathbb{N}$, $u_n > 0$, $v_n > 0$ and $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$ then the series associated to (u_n) is convergent.

c) If for all $n \in \mathbb{N}$, $u_n > 0$ and $\lim_{n\to\infty} \sqrt[n]{u_n} < 1$ then the series associated to (u_n) is convergent.

d) If for all $n \in \mathbb{N}$, $u_n > 0$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ then the series associated to (u_n) is convergent.

Exercise 7 Show that the series associated to the sequence $(\frac{1}{(n+1)^\alpha})$ is convergent when $\alpha > 1$. Hint: Compare $\sum_{\nu=0}^{n} \frac{1}{(\nu+1)^\alpha}$ with $1 + \int_1^{n+1} \frac{1}{t^\alpha} dt$.

4 Basic topology on $\mathbb{R}$

Definition 9 A subset F of $\mathbb{R}$ is closed if for all convergent sequences (u_n) such that $u_n \in F$ for all $n \in \mathbb{N}$, then the limit of (u_n) belongs to F.

A subset U of $\mathbb{R}$ is open if if for all convergent sequences (u_n) such that the limit belongs to U, then there exists $n_0 \in \mathbb{N}$ such that $u_n \in U$ for all $n \geq n_0$.

Remark 2 A closed interval is closed. An open interval is open. If $a < b$, the intervals $[a, b[$ and $]a, b]$ are neither open nor closed.

Exercise 8 Show that a singleton is a closed set. Show that a finite set is closed.

Proposition 12

A subset F of $\mathbb{R}$ is closed if and only if F^c, its complement in $\mathbb{R}$, is open.

A subset U of $\mathbb{R}$ is open if and only if U^c, its complement in $\mathbb{R}$, is closed.

A subset U of $\mathbb{R}$ is open if and only if for all $x \in U$, there exists $r > 0$ such that $]x - r, x + r[\subset U$.

Proposition 13

A finite union of closed sets is closed.

An intersection of finitely many or infinitely many closed sets is closed.

A finite intersection of open sets is open.

A union of finitely many or infinitely many open sets is open.

Definition 10 Let A be a subset of $\mathbb{R}$.

The closure of A is the set of real numbers ℓ such that there exists a sequence (u_n) converging to ℓ and satisfying $u_n \in A$ for all $n \in \mathbb{N}$. The closure of A is denoted cl$A$ or $\overline{A}$.

The interior of A is the set $a \in A$ for which there exists $r > 0$ such that $]a - r, a + r[\subset A$. The interior of A is denoted int$A$ or $\overset{\circ}{A}$.

Proposition 14 *Let A be a subset of $\mathbb{R}$.*

$A \subset \overline{A}$;

$\overline{A}$ is a closed subset of $\mathbb{R}$;

$\overline{A}$ is the smallest closed subset of $\mathbb{R}$ containing A, that is, if F is closed and $A \subset F$, then $\overline{A} \subset F$;

$\overline{A}$ is the intersection of all closed subsets of $\mathbb{R}$ containing A.

Proposition 15 *Let A be a subset of $\mathbb{R}$.*

$\text{int}A \subset A$;

$\text{int}A$ is an open subset of $\mathbb{R}$;

$\text{int}A$ is the largest open subset of $\mathbb{R}$ included in A, that is, if U is open and $U \subset A$, then $U \subset \text{int}A$;

$\overline{A}$ is the union of all open subsets of $\mathbb{R}$ included in A.

Exercise 9 Give the closure and the interior of the following subsets of $\mathbb{R}$.

$\mathbb{N}$;

$\mathbb{Q}$;

$\mathbb{R}$;

an interval of $\mathbb{R}$;

$\{\frac{1}{n+1} \mid n \in \mathbb{N}\}$;

$\{\frac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{0\}$.

Definition 11 Let A be a subset of $\mathbb{R}$. The boundary of A denoted $\text{bd}A$ is the set $\overline{A} \cap \overline{A^c}$, that is the intersection of the closure of A with the closure of the complement of A in $\mathbb{R}$.

Remark 3 An element b belongs to the boundary of A if and only if it is a limit of a sequence of elements of A and a limit of a sequence of elements not in A.

Proposition 16 *Let A be a subset of $\mathbb{R}$.*

The boundary of A is a closed set.

A is closed if and only if the boundary of A is included in A.

A is open if and only if the intersection of the boundary of A and A is empty, $\text{bd}A \cap A = \emptyset$.

Exercise 10 Give the boundary of the following subsets of $\mathbb{R}$.

$\mathbb{N}$;

$\mathbb{Q}$;

$\mathbb{R}$;

an interval of $\mathbb{R}$;

$\{\frac{1}{n+1} \mid n \in \mathbb{N}\}$;

$\{\frac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{0\}$.

Definition 12 Let A be a subset of $\mathbb{R}$. The set A is compact if it is closed and bounded.

Proposition 17 *Let A be a subset of $\mathbb{R}$. A is compact if one of the following equivalent conditions is satisfied:*

If (u_n) is a sequence such that $u_n \in A$ for all n, then it has a converging subsequence with a limit in A.

If $(U_i)_{i\in I}$ is a family of open subsets of $\mathbb{R}$ such that $A \subset \cup_{i\in I} U_i$, there exists a finite subset $J \subset I$ such that $A \subset \cup_{i\in J} U_i$.

If $(F_i)_{i\in I}$ is a family of closed subsets of $\mathbb{R}$ such that $A \cap (\cap_{i\in I} F_i) = \emptyset$, there exists a finite subset $J \subset I$ such that $A \cap (\cap_{i\in J} F_i) = \emptyset$.

5 Functions

In this section, I denotes an interval of $\mathbb{R}$ or a union of disjoint intervals. For example $\mathbb{R}$, $]0, +\infty[$, $[0, 1]$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\mathbb{R} \setminus \mathbb{N}$.

Definition 13 Let f be a function from I to $\mathbb{R}$.

f is bounded if there exists $r > 0$ such that for all $x \in I$, $f(x) \in [-r, r]$.

The image of I by f is the set $\{y \in \mathbb{R} \mid \exists x \in I, y = f(x)\}$.

An element $\bar{x} \in I$ is a maximum (resp. a minimum) of f on I if for all $x \in I$, $f(x) \leq$ (resp. $\geq$)$f(\bar{x})$.

Limit of a function

Definition 14 Let f be a function from I to $\mathbb{R}$. Let x_0 an element of the closure of I.

The function f has a limit y_0 at x_0 if for all sequences (u_n) satisfying $u_n \in I$ for all n and $\lim_{n\to\infty} u_n = x_0$, then the sequence $(f(u_n))$ is convergent and its limit is y_0.

The function f has a right (resp. left) limit y_0 at x_0 if for all sequences (u_n) satisfying $u_n \in I$ and $u_n \geq$ (resp. $\leq$)x_0 for all n and $\lim_{n\to\infty} u_n = x_0$, then the sequence $(f(u_n))$ is convergent and its limit is y_0.

Proposition 18 *Let f be a function from I to $\mathbb{R}$. Let x_0 an element of the closure of I.*

The function f has at most one limit at x_0.

If f has a right and a left limit at x_0, then f has a limit at x_0 if and only if the right and the left limits are equal.

The function f has a limit y_0 at x_0 if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in]x_0 - \rho, x_0 + \rho[\cap I$, $f(x) \in]y_0 - r, y_0 + r[$.

Cauchy criterion: the function f has a limit at x_0 if and only if for all $r > 0$, there exists $\rho > 0$ such that for all pair (x, x') in $]x_0 - \rho, x_0 + \rho[\cap I$, $|f(x) - f(x')| < r$.

Limit at infinity

Definition 15 Let f be a function from I to $\mathbb{R}$. We assume that I is not bounded above (resp. below). The function f has a limit y_0 at $+\infty$ (resp. $-\infty$) if for all sequences (u_n) satisfying $u_n \in I$ for all n and $\lim_{n\to\infty} u_n = +\infty(\text{resp.} -\infty)$, then the sequence $(f(u_n))$ is convergent and its limit is y_0.

Proposition 19 *Let f be a function from I to $\mathbb{R}$. We assume that I is not bounded above (resp. below).*

The function f has a limit y_0 at at $+\infty$ (resp. $-\infty$) if for all $r > 0$, there exists $\rho \in \mathbb{R}$ such that for all $x \in]\rho, +\infty[\cap I$ (resp. $]-\infty, \rho[\cap I$), $f(x) \in]y_0 - r, y_0 + r[$.

Infinite limits

Definition 16 (i) Let f be a function from I to $\mathbb{R}$. Let $x_0 \in \overline{I}$.

The function f tends to $+\infty$ (resp. $-\infty$) at x_0 if for all sequences (u_n) satisfying $u_n \in I$ for all n and $\lim_{n\to\infty} u_n = x_0$, then the sequence $(f(u_n))$ tends to $+\infty$ (resp. $-\infty$). Equivalently, the function f tends to $+\infty$ (resp. $-\infty$) at x_0 if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in]x_0 - \rho, x_0 + \rho[\cap I$, $f(x) \in]r, +\infty[$ (resp. $]-\infty, r[$).

(ii) Let f be a function from I to $\mathbb{R}$. We assume that I is not bounded above.

The function f tends to $+\infty$ (resp. $-\infty$) at $+\infty$ if for all sequences (u_n) satisfying $u_n \in I$ for all n and $\lim_{n\to\infty} u_n = +\infty$, then the sequence $(f(u_n))$ tends to $+\infty$ (resp. $-\infty$). Equivalently, the function f tends to $+\infty$ (resp. $-\infty$) at $+\infty$ if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in]\rho, +\infty[\cap I$, $f(x) \in]r, +\infty[$ (resp. $]-\infty, r[$).

(iii) Let f be a function from I to $\mathbb{R}$. We assume that I is not bounded below.

The function f tends to $+\infty$ (resp. $-\infty$) at $-\infty$ if for all sequences (u_n) satisfying $u_n \in I$ for all n and $\lim_{n\to\infty} u_n = -\infty$, then the sequence $(f(u_n))$

tends to $+\infty$ (resp. $-\infty$). Equivalently, the function f tends to $+\infty$ (resp. $-\infty$) at $-\infty$ if for all $r > 0$, there exists $\rho > 0$ such that for all $x \in]-\infty, \rho[\cap I$, $f(x) \in]r, +\infty[$ (resp. $]-\infty, r[$).

Notations We denotes the limit at x_0 by $\lim_{x \to x_0, x \in I} f(x)$ assuming that the limit could be infinite. If the domain of definition I is clearly defined, we often omit $x \in I$ and we denote the limit $\lim_{x \to x_0} f(x)$. The right limit is denoted $\lim_{x \to x_0^+, x \in I} f(x)$ and the left limit $\lim_{x \to x_0^-, x \in I} f(x)$.

For the limit at infinity, we denote it by $\lim_{x \to +\infty, x \in I} f(x)$ or $\lim_{x \to -\infty, x \in I} f(x)$.

Inequalities and limits

Proposition 20 *Let f and g be two functions from I to $\mathbb{R}$. Let $x_0 \in \overline{I}$. x_0 could be $+\infty$ if I is not bounded above or $-\infty$ if I is not bounded below. We assume that f and g have a limit at x_0. Then*

a) If for all $x \in I$, $f(x) \leq g(x)$, then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$.

b) If there exists a real number $m \in \mathbb{R}$ such that for all $x \in I$, $f(x) \leq m$, then $\lim_{x \to x_0} f(x) \leq m$.

c) If there exists a real number $m \in \mathbb{R}$ such that for all $x \in I$, $f(x) \geq m$, then $\lim_{x \to x_0} f(x) \geq m$.

Basic calculus with limits

Proposition 21 *Let f and g be two functions from I to $\mathbb{R}$. Let $x_0 \in \overline{I}$. We assume that f and g have a finite limit at x_0 denoted y_0 and z_0. Then*

a) The function $f + g$ has a limit at x_0 which is $y_0 + z_0$.

b) The function fg has a limit at x_0 which is $y_0 z_0$.

c) For all $t \in \mathbb{R}$, the function tf has a limit at x_0 which is ty_0. In particular, $\lim_{x \to x_0} -f(x) = -\lim_{x \to x_0} f(x)$.

d) The function $|f|$ has a limit at x_0 which is $|y_0|$

e)The function $\max\{f, g\}$ has a limit at x_0 which is $\max\{y_0, z_0\}$.

f) The function $\min\{f, g\}$ has a limit at x_0 which is $\min\{y_0, z_0\}$.

g) If $z_0 \neq 0$ then there exists $r > 0$ such that the function $\frac{f}{g}$ is defined on $]x_0 - r, x_0 + r[\cap I$ and it has a limit at x_0 which is $\frac{y_0}{z_0}$.

Remark 4 The above results holds true if I is not bounded above (resp. bouded below), for the limit at $+\infty$ (resp. $-\infty$).

Calculus rules with infinite limits Let f and g be two functions from I to $\mathbb{R}$. Let $x_0 \in \overline{I}$. x_0 could be $+\infty$ if I is not bounded above or $-\infty$ if I is not bounded below. We assume that f and g have a finite or infinite limit at x_0 denoted ℓ and ℓ'.

$\lim_{x \to x_0}(f+g)(x)$

$\lim_{x\to x_0} f(x)$ / $\lim_{x\to x_0} g(x)$	$-\infty$	ℓ	$+\infty$
$-\infty$	$-\infty$	$-\infty$	?
ℓ'	$-\infty$	$\ell+\ell'$	$+\infty$
$+\infty$	?	$+\infty$	$+\infty$

$\lim_{x \to x_0}(fg)(x)$

$\lim_{x\to x_0} f(x)$ / $\lim_{x\to x_0} g(x)$	$-\infty$	$\ell<0$	0	$\ell>0$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	?	$-\infty$	$-\infty$
$\ell'<0$	$+\infty$	$\ell\ell'$	0	$\ell\ell'$	$-\infty$
0	?	0	0	0	?
$\ell'>0$	$-\infty$	$\ell\ell'$	0	$\ell\ell'$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	?	$+\infty$	$+\infty$

Limite of the composition of two functions

Proposition 22 *Let f be a function on I and $x_0 \in \overline{I}$. x_0 could be $+\infty$ if I is not bounded above or $-\infty$ if I is not bounded below. Let g be a function on J. We assume that for all $x \in I$, $f(x) \in J$. Let $y_0 = \lim_{x\to x_0} f(x)$. One easily checks that $y_0 \in \overline{J}$ or $y_0 = +\infty(resp. -\infty)$ and J is not bounded above (resp. below). Let $z_0 = \lim_{y\to y_0} g(y)$. Then the limit of $g \circ f$ at x_0 exists and it is equal to z_0.*

Continuous functions

Definition 17 Let f be a function from I to $\mathbb{R}$. f is continuous at a point $x_0 \in I$, if the limit of f at x_0 exists and is equal to $f(x_0)$. f is continuous on I if f is continuous at every point of I.

Remark 5 All the usual functions: absolute value function, polynomial, fraction of polynomial, logarithm, exponential, trigonometric functions are continuous on their domain of definition.

A particular class of continuous function is the class of **Lipschitzian** functions, that is the function f from I to $\mathbb{R}$ such that there exists $k \geq 0$, for all $(x,x') \in I \times I$, $|f(x)-f(x')| \leq k|x-x'|$.

Proposition 23 *Let f be a function from I to $\mathbb{R}$. f is continuous on I if one of the two equivalent following conditions is satisfied:*

For all open set U of $\mathbb{R}$, the set $f^{-1}(U) = \{x \in I \mid f(x) \in U\} = V \cap I$ where V is an open set of $\mathbb{R}$.

For all closed set F of $\mathbb{R}$, the set $f^{-1}(F) = \{x \in I \mid f(x) \in F\} = G \cap I$ where G is a closed set of $\mathbb{R}$.

Proposition 24 *Let f be a continuous function from I to $\mathbb{R}$. Then*

$|f|$ is a continuous function from I to $\mathbb{R}$.

for all $t \in \mathbb{R}$, tf is a continuous function from I to $\mathbb{R}$.

Proposition 25 *Let f and g be two continuous functions from I to $\mathbb{R}$. Then*

$f + g$ is a continuous function from I to $\mathbb{R}$.

fg is a continuous function from I to $\mathbb{R}$.

if $g(x) \neq 0$ for all $x \in I$, $\dfrac{f}{g}$ is a continuous function from I to $\mathbb{R}$.

Proposition 26 *Let f be a continuous function on I. Let g be a continuous function on J. We assume that for all $x \in I$, $f(x) \in J$. Then $g \circ f$ is continuous on I.*

With these basic operations, we are able to show almost always that the usual functions are continuous.

Remark 6 If f is continuous on I and has a limit at $x_0 \in \overline{I} \setminus I$, then the mapping $\tilde{f}$ from $J = I \cup \{x_0\}$ to $\mathbb{R}$ defined by $\tilde{f}(x) = f(x)$ if $x \in I$ and $\tilde{f}(x_0) = \lim_{x \to x_0, x \in I} f(x)$ is a continuous function on J called the continuous extension of f. If a continuous extension of f exists on $\overline{I}$, it is unique.

The following criterion of continuity is very useful when the function f is defined as a solution of an optimisation problem.

Proposition 27 *Let f a* **bounded** *function from I to $\mathbb{R}$. Then f is continuous on I if and only if the graph of f is closed, that is, for all sequences (x_n) of elements of I converging to $x_0 \in I$ and the sequence $(f(x_n))$ converges in $\mathbb{R}$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.*

Exercise 11 Let f be a bounded continuous function from $\mathbb{R}$ to $\mathbb{R}$. Show that the function $g(y) = \sup_{x \in [-y,y]}\{f(x)\}$ is continuous on $\mathbb{R}_+$.

5.1 Continuous function on a closed interval

Theorem 2 *Let $I = [a, b]$ be a closed interval and f a continuous function from I to $\mathbb{R}$. Then $f([a, b])$ is a closed interval.*

Corollary 1 *Intermediate Value Theorem. Let $I = [a, b]$ be a closed interval and f a continuous function from I to $\mathbb{R}$. Let $(c, d) \in [a, b] \times [a, b]$ with $f(c) \leq f(d)$. Then, for all $y \in [f(c), f(d)]$, there exists $x \in I$ such that $f(x) = y$.*

Corollary 2 *Weierstrass Theorem. Let $I = [a, b]$ be a closed interval and f a continuous function from I to $\mathbb{R}$. Then there exists $\overline{x} \in I$ and $\underline{x} \in I$ such that for all $x \in I$, $f(\underline{x}) \leq f(x) \leq f(\overline{x})$.*

Corollary 3 *Weierstrass Theorem. Let A be a bounded closed (compact) subset of $\mathbb{R}$ and f a continuous function from A to $\mathbb{R}$. Then there exists $\overline{x} \in A$ and $\underline{x} \in A$ such that for all $x \in A$, $f(\underline{x}) \leq f(x) \leq f(\overline{x})$.*

Theorem 3 *Let f be continuous and strictly monotone function from $[a, b]$ to $f([a, b])$. Then f is onto and one to one and f^{-1} is continuous.*

Banach fixed point theorem

Theorem 4 *Let f be a function from an* **interval** *I to $\mathbb{R}$. We assume that $f(I) \subset I$ and f is a contraction, that is, there exists $k \in [0, 1[$ such that for all $(x, x') \in I \times I$, $|f(x) - f(x')| \leq k|x - x'|$. Then there exists a unique element (fixed point) $\bar{x} \in I$ such that $f(\bar{x}) = \bar{x}$ and for all $x_0 \in I$, the sequence (u_n) defined by $u_0 = x_0$ and for all $n \in \mathbb{N}$, $u_{n+1} = f(u_n)$ converges to $\bar{x}$.*

Sequence of continuous bounded functions

Let I be an interval of $\mathbb{R}$ or a union of disjoint intervals. Let (f_n) be a sequence of bounded continuous function on I. We assume that for all $x \in I$, the real sequence $(f_n(x))$ is convergent. So we can define a function f on I by $f(x) = \lim_{n \to \infty} f_n(x)$. The question is to find a sufficient condition to obtain the continuity of f as a function from I to $\mathbb{R}$.

Theorem 5 *If f is bounded and the real sequence $(\sup_{x \in I} \{|f_n(x) - f(x)|\})$ converges to 0, then f is continuous on I. In this case, we say that the sequence (f_n) converges uniformly to f.*

Like for the real sequences, we have a Cauchy criterion for the uniform convergence of limit of continuous functions.

Theorem 6 *Let I be an interval of $\mathbb{R}$ or a union of disjoint intervals. Let (f_n) be a sequence of bounded continuous function on I. If for all $r > 0$, there exists $n \in \mathbb{N}$ such that for all $p, q \geq n$, $\sup_{x \in I} \{|f_p(x) - f_q(x)|\} \leq r$, then there exists a continuous function f on I such that $\lim_{n \to \infty} \sup_{x \in I} \{|f_n(x) - f(x)|\} = 0$, which implies that for all $x \in I$, $f(x) = \lim_{n \to \infty} f_n(x)$.*

5.2 Derivative

Let f be a function defined on I, which is an interval or a union of disjoint intervals. Let x and y two different points of I. The rate of increasing of f

between x and y is the ratio $\frac{f(y)-f(x)}{y-x}$, which is the slope of the line joining the two points of the graph of f, $(x, f(x))$ and $(y, f(y))$. The derivative of f at x is the limit of this rate of increasing when y tends to x.

Definition 18 Let f be a function defined on I and x_0 be an element of interior of I. Then f is differentiable at x_0 if the following limit exists:

$$\lim_{x \to x_0, x \in I, x \neq x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This limit is called the derivative of f at x_0. It is denoted $f'(x_0)$.

If there exists $r > 0$ such that $]x_0 - r, x_0] \subset I$, f is left differentiable at x_0 if the following limit exists:

$$\lim_{x \to x_0^-, x \in I, x \neq x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This limit is called the left derivative of f at x_0. It is denoted $f_l'(x_0)$.

If there exists $r > 0$ such that $[x_0, x_0 + r[\subset I$, f is right differentiable at x_0 if the following limit exists:

$$\lim_{x \to x_0^+, x \in I, x \neq x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This limit is called the right derivative of f at x_0. It is denoted $f_r'(x_0)$.

If I is open, f is differentiable on I if f is differentiable at each point of I. f is continuously differentiable on I if the function $x \to f'(x)$ is continuous on I.

If I is a closed bounded interval $[a, b]$, f is differentiable on I if it is differentiable on $]a, b[$ and it has a right derivative at a and a left derivative at b. f is continuously differentiable on I if the function f' extended by $f_r'(a)$ and $f_l'(b)$ is continuous

Exercise 12 Compute the derivative of f for the following functions if it exists at all point of its domain of definition.

$f(x) = m$ where m is a real number;

$f(x) = x$;

$f(x) = |x|$;

$f(x) = x^2$;

$f(x) = \sqrt{x}$;

Proposition 28 *If f is differentiable at a point x_0 in its domain, it is continuous at x_0.*

Proposition 29 *Let f and g be two functions defined on I and differentiable at $x_0 \in I$. Then*

$f+g$ is differentiable at x_0 and the derivative is $f'(x_0)+g'(x_0)$;

fg is differentiable at x_0 and the derivative is $f'(x_0)g(x_0)+g'(x_0)f(x_0)$;

if $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and the derivative is $\frac{f'(x_0)g(x_0)-g'(x_0)f(x_0)}{(g(x_0))^2}$;

Proposition 30 *Let f be a function on I differentiable at $x_0 \in I$. Let g be a function defined on J. We assume that $f(x_0) \in \mathrm{int}J$ and g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and the derivative is $f'(x_0)g'(f(x_0))$.*

Corollary 4 *If f is differentiable on I an* **interval** *of $\mathbb{R}$ and $f'(x) \geq$ (resp. $>$)0 for all $x \in I$, then f is (resp. strictly) increasing on I.*

If f is differentiable on I an **interval** *of $\mathbb{R}$ and $f'(x) \leq$ (resp. $<$)0 for all $x \in I$, then f is (resp. strictly) decreasing on I.*

Proposition 31 *Let f be a differentiable mapping on I an open* **interval** *of $\mathbb{R}$. Let us assume that $f'(x) > 0$ for all $x \in I$. Then f is one-to-one and onto from I to $f(I)$ and the inverse mapping f^{-1} is differentiable on $f(I)$ and its derivative is $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.*

A similar result holds true if $f'(x) < 0$ for all $x \in I$.

Theorem 7 *Rolle's Theorem and Mean value theorem Let f be a continuous function on $[a,b]$, differentiable on $]a,b[$ then there exists $c \in\,]a,b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

In particular, if $f(a) = f(b)$, then there exists $c \in]a,b[$ such that $f'(c) = 0$.

Corollary 5 *Let f be a continuous function on $[a,b]$, differentiable on $]a,b[$.*

If $f'(x) \geq 0$ for all $x \in]a,b[$, then f is increasing on $[a,b]$;

If $f'(x) > 0$ for all $x \in]a,b[$, then f is strictly increasing on $[a,b]$;

If $f'(x) \leq 0$ for all $x \in]a,b[$, then f is decreasing on $[a,b]$;

If $f'(x) < 0$ for all $x \in]a,b[$, then f is strictly decreasing on $[a,b]$;

Corollary 6 *Let f be a continuous function on $[a,b]$, differentiable on $]a,b[$. If there exists a non-negative real number k such that $|f'(x)| \leq k$ for all $x \in\,]a,b[$, then f is Lipschitz continuous on $[a,b]$ of rank k, that is, for all $(x,x') \in [a,b] \times [a,b]$, $|f(x) - f(x')| \leq k|x - x'|$.*

6 Taylor development of a function

6.1 Higher order derivatives

Let $]a, b[$ be an interval with $a < b$. Let f be a function from $]a, b[$ to $\mathbb{R}$.

The second derivative f'' is the derivative of the derivative and the derivative of order $p + 1$ is the derivative of the derivative of order p. It is denoted $f^{(p+1)}$. We can also define the left and right derivatives of higher order when f is defined on $]x_0 - r, x_0]$ or $[x_0, x_0 + r[$.

The same formulas apply for the computation of the higher order derivatives. For the computation of the derivative of order p of a product of two functions, we have the following *Leibniz* formula:

$$(fg)^{(p)} = f^{(p)}g + C_p^1 f^{(p-1)}g' + C_p^2 f^{(p-2)}g'' + \ldots + C_p^k f^{(p-k)}g^{(k)} \ldots + fg^{(p)}$$

where C_p^k is the binomial coefficient, that is $C_p^k = \frac{p!}{k!(p-k)!}$.

Let p be an integer greater of equal to 1. f is of class $\mathcal{C}^p$ on $]a, b[$ if$f$ has derivatives until order p on $]a, b[$ and $f^{(p)}$ is continuous. This implies that all derivatives until order p of f and f are continuous. f is of class $\mathcal{C}^\infty$ if f has derivatives for all order $p \in \mathbb{N}^*$. This implies that all derivatives of f and f are continuous. We can also define function of class $\mathcal{C}^p$ or $\mathcal{C}^\infty$ on a closed interval $[a, b]$ by considering the right derivatives at a and the left derivatives at b to extend the higher order derivatives from $]a, b[$ to $[a, b]$.

6.2 Taylor Formula

Let f of class $\mathcal{C}^p$ on $[a, b]$ such that $f^{(p+1)}$ exists on $]a, b[$. Then, there exists $c \in]a, b[$ such that:

$$f(b) = f(a) + (b - a)f'(a) + \ldots + \frac{(b - a)^p}{p!} f^{(p)}(a) + \frac{(b - a)^{p+1}}{(p + 1)!} f^{(p+1)}(c)$$

For all $x \in [a, b]$, there exists $\theta \in]0, 1[$ such that:

$$f(x) = f(a) + (x - a)f'(a) + \ldots + \frac{(x - a)^p}{p!} f^{(p)}(a) + \frac{(x - a)^{p+1}}{(p + 1)!} f^{(p+1)}(a + \theta(x - a))$$

If $|f^{(p+1)}(x)|$ is upper bounded on $]a, b[$ by M, then for all $x \in [a, b]$,

$$|f(x) - (f(a) + (x - a)f'(a) + \ldots + \frac{(x - a)^p}{p!} f^{(p)}(a))| \leq M \frac{(x - a)^{p+1}}{(p + 1)!}$$

Remark 7 When $a = 0$, we obtain the following formula: for all $x \in [0, b]$, there exists $\theta \in]0, 1[$ such that:

$$f(x) = f(a) + xf'(a) + \ldots + \frac{x^p}{p!} f^{(p)}(a) + \frac{x^{p+1}}{(p + 1)!} f^{(p+1)}(\theta x)$$

Development of Taylor-Young

Let f of class $\mathcal{C}^p$ on $[a,b]$ such that $f_r^{(p+1}(a)$ exists. Then, there exists a function η from $[a,b]$ to $\mathbb{R}$ such that $\lim_{x\to a}\eta(x)=0$ and for all $x\in[a,b]$,

$$f(x)=f(a)+(x-a)f'(a)+\ldots+\frac{(x-a)^p}{p!}f^{(p)}(a)+\frac{(x-a)^{p+1}}{(p+1)!}f^{(p+1)}(a)+(x-a)^{p+1}\eta(x)$$

Taylor Development with an integral

Let f of class $\mathcal{C}^{p+1}$ on $[a,b]$. Then, for all $x\in[a,b]$,

$$f(x)=f(a)+(x-a)f'(a)+\ldots+\frac{(x-a)^p}{p!}f^{(p)}(a)+\frac{1}{p!}\int_a^x(x-t)^pf^{(p+1)}(t)dt$$

6.3 Power serie

Let (a_n) be a sequence such that $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ is finite. Then, let $r=1/\lim_{n\to\infty}\sqrt[n]{|a_n|}$ (with $r=+\infty$ if the limit is 0). From the convergence criteria for series (See Proposition 11 (c)), for all $x\in]-r,r[$, the series associated to the sequence (a_nx^n) is convergent, that is, $\lim_{n\to\infty}\sum_{\nu=0}^n a_\nu x^\nu$ exists. So, we can define a function φ on $]-r,r[$ by:

$$\varphi(x)=\lim_{n\to\infty}\sum_{\nu=0}^n a_\nu x^\nu=\sum_{\nu=0}^\infty a_\nu x^\nu$$

In other words, $\varphi(x)$ is the limit of the polynomial $(P_n(x)=\sum_{\nu=0}^n a_\nu x^\nu)$.

The key properties of φ are sumarized below.

Proposition 32 *a) φ is the uniform limit of $(P_n(x))$ on all closed segments $[-\rho,\rho]$ for all $\rho<r$.*

b)φ is of class $\mathcal{C}^\infty$ on $]r,r[$.

c)The p derivative of φ is given by the following formula:

$$\varphi^{(p)}=\sum_{\nu=0}^\infty(\nu+p)(\nu+p-1)\ldots(\nu+1)a_\nu x^\nu$$

d) In particular $\varphi^{(p)}(0)=p!a_p$.

Definition 19 A function f defined on an open interval $]-r,r[$ is analytic if it is equal to $\sum_{\nu=0}^\infty\frac{f^{(\nu)}(0)}{\nu!}x^\nu$ with $f^{(0)}=f$ and $0!=1$ for all $x\in]-r,r[$.

Remark 8 All usual functions are analytic. We can also defined analytic functions around x_0 be considering the change of variable $t=x-x_0$.

Remark 9 For two analytic functions f and g on the same interval $]-r, r[$, we can extend the usual formula applied to polynomials. For example, $f+g$ is analytic and

$$(f+g)(x) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0) + g^{(\nu)}(0)}{\nu!} x^{\nu}$$

fg is analytic and

$$fg(x) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0)g(0) + C_{\nu}^{1} f^{(\nu-1)}(0)g'(0) + C_{\nu}^{2} f^{(\nu-2)}(0)g''(0) + \ldots + f(0)g^{(\nu)}(0)}{\nu!} x^{\nu}$$

7 Usual functions on $\mathbb{R}$

Usual limits

Limit at $-\infty$ and $+\infty$ of a polynomial function.
$a > 0$ and $b \in \mathbb{R}$
$\lim_{x\to+\infty} \frac{(\ln x)^b}{x^a} = 0$
$\lim_{x\to 0^+} x^a (\ln x)^b = 0$
$\lim_{x\to+\infty} \frac{(\exp x)^a}{x^b} = +\infty$
$\lim_{x\to-\infty} (\exp x)^a x^b = 0$

Polynomial

Domain: $\mathbb{R}$
$f(x) = a_0 + a_1 x + \ldots + a_p x^p$
derivative $f'(x) = a_1 + 2a_2 x + \ldots + p a_p x^{p-1}$
primitive $F(x) = c + a_0 x + \frac{a_1}{2} x^2 + \ldots + \frac{a_p}{p+1} x^{p+1}$

Power

Domain: $\mathbb{R}_+$ or $\mathbb{R}_+^*$
$f(x) = x^a$
derivative $f'(x) = a x^{a-1}$
primitive $F(x) = c + \frac{1}{a+1} x^{a+1}$ si $a \neq -1$, $c + \ln x$ si $a = -1$

exponential

Domain: $\mathbb{R}$
$f(x) = e^x$
derivative $f'(x) = e^x$
primitive $F(x) = c + e^x$

Remark 10 $a > 0$, $a^x = e^{x \ln a}$, derivative $\ln a e^{x \ln a}$, primitive $c + \frac{1}{\ln a} e^{x \ln a}$ if $a \neq 1$, $c + x$ otherwise.

logarithm

Domain: $\mathbb{R}_+^*$
$f(x) = \ln x$
derivative $f'(x) = \frac{1}{x}$

primitive $F(x) = c + x(\ln x - 1)$

Remark 11 $a > 0, a \neq 1$, the logarithm with base a is $\log_a x = \frac{\ln x}{\ln a}$

sinus

Domain: $\mathbb{R}$
$f(x) = \sin x$
derivative $f'(x) = \cos x$
primitive $F(x) = c - \cos x$

cosine

Domain: $\mathbb{R}$
$f(x) = \cos x$
derivative $f'(x) = -\sin x$
primitive $F(x) = c + \sin x$

tangent

Domain: $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$
$f(x) = \frac{\sin x}{cosx}$
derivative $f'(x) = \frac{1}{\cos^2 x}$
primitive $F(x) = c - \ln(\cos x)$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$

arcsine

Domain: $[-1, 1]$
$f(x) = \arcsin x$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$
derivative $f'(x) = \frac{1}{\sqrt{1-x^2}}$
Primitive $F(x) = c + x \arcsin x + \sqrt{1 - x^2}$

arccosine

Domain: $[-1, 1]$
$f(x) = \arccos x$ in $[0, \pi]$
derivative $f'(x) = \frac{-1}{\sqrt{1-x^2}}$
Primitive $F(x) = c + x \arccos x - \sqrt{1 - x^2}$

arctangent

Domain: $\mathbb{R}$
$f(x) = \arctan x$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$
derivative $f'(x) = \frac{1}{1+x^2}$
Primitive $F(x) = c + x \arctan x - \frac{1}{2} \ln(1 + x^2)$