Université Paris 1 Panthéon-Sorbonne

Lecture notes

Master Mathematical Models in Economics and Finance (MMEF)

LOGIC AND SETS

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  and Xavier Venel `https://sites.google.com/site/xaviervenelsite/course-materials`
- A first Course in Mathematical Logic and Set Theory, Michael L. O'Leary.
- Schaum's Outline of Set Theory and Related Topics, Seymour Lipschutz

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Chapter

1 Classical logic

1.1 Symbolic logic

Definition 1. A proposition is a statement which is either true or false.

Example 1.

- "Paris is in France" or "2+2=4" (are true).
- "2+2=5" (is false).
- "What time is it?" is not a proposition.

Logic is all about propositions and relationships between them.

Definition 2. Let p and q be two propositions:

(i) $p \wedge q$, called the conjunction of p and q, is the proposition which is true if and only if p is true and q is true.

(ii) $p \vee q$, called the disjunction of p and q, is the proposition which is true if p is true or q is true.

(iii) $\neg p$, called the negation of p, is the proposition which is true if and only if p is false.

(iv) The material implication $p \to q$, "if p then q", is the abbreviation of the proposition $\neg p \vee q$ (which is true unless p is true and q is false).

(v) The material equivalence $p \leftrightarrow q$, "p if and only if q", is the abbreviation of the proposition $(p \to q) \wedge (q \to p)$.

We can use a truth value tabular to evaluate if a "complex" proposition is true or false. For example :

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$p \leftrightarrow q$	$p \vee \neg p$	$p \wedge \neg p$
T	T	T	T	F	F	T	T	T	T	F
T	F	F	T	F	T	F	T	F	T	F
F	T	F	T	T	F	T	F	F	T	F
F	F	F	F	T	T	T	T	T	T	F

Definition 3. A tautology is a proposition which is always true regardless of which valuation is used for the propositional variables.

Definition 4. The sentence "if p then q" or "p implies q" or "p is a sufficient condition for q" or "q is a necessary condition for p" or "p only if q" is denoted by the abbreviation: "$p \Rightarrow q$" which means:

"$p \to q$" is true.

Example 2. Let p and q be the two following propositions:

- p: "1=2" (false)
- q: "2=3" (false)

$p \to q$ is true (see truth table). Hence we can write $p \Rightarrow q$.

Definition 5. The sentence "p if and only if q" or "p is equivalent to q" or "p is a necessary and sufficient condition for q" is denoted by the abbreviation: "$p \Leftrightarrow q$" which means:

"$p \leftrightarrow q$" is true.

Theorem 1. Let p, q and r be three propositions. The following propositions are tautologies:

(i) $p \vee (\neg p)$

(ii) $p \leftrightarrow (\neg(\neg p))$

(iii) $(p \vee q) \leftrightarrow (q \vee p)$

(iv) $(p \wedge q) \leftrightarrow (q \wedge p)$

(v) $((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$

(vi) $((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$

(vii) $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

(viii) $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$

(ix) $(p \vee q) \wedge r) \leftrightarrow (p \wedge r) \vee (q \wedge r)$

(x) $(p \wedge q) \vee r) \leftrightarrow (p \vee r) \wedge (q \vee r)$

Exercice : Find the negation of the proposition

"If there is a problem then there is a solution".

Solution :

"There is a problem and there is no solution"

hint: use Theorem 1(vii) and (ii) with the propositions p:"there is a problem", and q:"there is a solution".

1.2 First-order logic

We gave a mathematical formalism for the sentence: "Tuesday is a nice day". We would like now to construct from this sentence "Every day of the week is a nice day".

Definition 6 (Naive definition of a set)**.** A set is a well-defined collection of objects. The objects in a set are called its elements. If A is a set,

(i) $a \in A$ means that a belongs to the set A, or that a is an element of A, or that A contains a.

(ii) $a \notin A$ means that a does not belong to A or that a is not an element of A, or that A does not contain a.

Example 3. The easiest way to define a set is to list all elements. For example, $\{\alpha, \beta, \gamma\}$ is the set which contains the elements called α, β and γ.

Definition 7. A predicate on a set A is an expression which associates to every element $x \in A$ a proposition $p(x)$.

Definition 8 (Universal Quantifier)**.** Given the predicate $p(x)$ on A, "$\forall x \in A, p(x)$" is the proposition which is true if and only if $p(x)$ is true for every element $x \in A$.

Example 4.

- $\forall x \in \mathbb{R},\ x^2 \geq 0$,
- $\forall x \in \mathbb{R},\ x + 1 = 2$.

Definition 9 (Existential Quantifier)**.** Given the predicate $p(x)$ on A, "$\exists x \in A, p(x)$" is the proposition which is true if and only if $p(a)$ is true for at least one element $a \in A$.

Example 5. Let $p(\alpha), p(\beta)$ and $p(\gamma)$ be three propositions which define a predicate on the set $\{\alpha, \beta, \gamma\}$. We have "$\exists x \in \{\alpha, \beta, \gamma\}, p(x)$" if and only if $p(\alpha) \vee p(\beta) \vee p(\gamma)$.

Definition 10 (Negation of quantified statements)**.** The negation of quantified statements is based on the following rule:

(i) $\neg(\forall x \in A, p(x)) \Leftrightarrow (\exists x \in A, \neg p(x))$

(ii) $\neg(\exists x \in A, p(x)) \Leftrightarrow (\forall x \in A, \neg p(x))$

Example 6. Let $A = \{$Lucy, Matthew, Lisa$\}$ and $P(x)$ the predicate on A, "x wears a red shirt". Let us denote by $p =$"Lucy wears a red shirt", $q =$"Matthew wears a red shirt", $r =$"Lisa wears a red shirt", then we have

p	q	r	$\forall x \in A, P(x)$	$\neg(\forall x \in A, p(x))$	$\neg p$	$\neg q$	$\neg r$	$\exists x \in A, (\neg p(x))$
T	T	T	T	F	F	F	F	F
T	T	F	F	T	F	F	T	T
T	F	T	F	T	F	T	F	T
T	F	F	F	T	F	T	T	T
F	T	T	F	T	T	F	F	T
F	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	F	T
F	F	F	F	T	T	T	T	T

Definition 11. A predicate can concern several variables. Let p be a predicate on $A \times B$ and Q_A and Q_B be two quantifiers then

- $Q_A\ x \in A,\ p(x, y)$ is a predicate on B,
- $Q_B\ y \in B,\ Q_A x \in A,\ p(x, y)$ is a proposition.

Example 7.

- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x \leq y$,
- $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x \leq y$.

Proposition 1. Let p be a predicate on $A \times B$ then

$$\exists y \in B,\ \forall x \in A, p(x, y) \Rightarrow \forall x \in A, \exists y \in B,\ p(x, y).$$

Remark 1. The order of quantifiers (if of different types) is important. If they are all of the same type the order is not important.

Remark 2. (Negation of complex quantifiers) Let p be a predicate on $A \times B$, then

$$\begin{aligned} \neg\left(\forall x \in A,\ \exists y \in B, p(x, y)\right) &\Leftrightarrow \exists x \in A,\ \neg(\exists y \in B,\ p(x, y)), \\ &\Leftrightarrow \exists x \in A,\ \forall y \in B,\ \neg(p(x, y)). \end{aligned}$$

Example 8. Let f be a function on an interval I, the function is said to be continuous on I if

$$\forall x \in I, \forall \varepsilon > 0, \exists \delta > 0,\ \text{s.t. } \forall y \in]x - \delta, x + \delta[, |f(x) - f(y)| \leq \varepsilon.$$

By taking the negation and applying successively the rules of inference on the negations: f is not continuous on I if

$$\exists x \in I, \exists \varepsilon > 0, \forall \delta > 0,\ \text{s.t. } \exists y \in]x - \delta, x + \delta[, |f(x) - f(y)| > \varepsilon.$$

Chapter

2 Reasoning in mathematics

In this chapter, we will highlight several typical ways to prove a mathematical statement. The type of proof will depend on the statement.

The main statements that you can find are of the following form:

- Universal statement: For every .....,... (is true).
- Conditional statement: If ....(is true) then .... (is true).
- Existential statement: There exists ... such that ... (is true).
- Equivalence statement: ...(is true) if and only if ... (is true).

Remark 3. Universal statements and conditional statements are in fact very similar: "Every even number is followed by an odd number" is equivalent to "If n is an even number then it is followed by an odd number" (the quantifier is omitted in this last formulation).

Remark 4. In the process of writing a proof there are several levels of formalism: the intuition (where anything is allowed), a sketch of the proof with the order of the arguments, a detailed formal proof where each step is written properly.

2.1 Existential/Conditional statements and direct proof

We start by introducing direct proofs. The intuition is to follow the direction of the statement. The precise way will depend on the type of statements (conditional, existential or equivalence).

Our example will concern integers (denoted by $\mathbb{N}$), real numbers and divisibility. We first introduce the definitions that we will use.

Definition 12. Let p and n be two integers. We say that p is *dividing* n or that p is *a divisor of* n if there exists an integer k such that $n = kp$.

$$\exists k \in \mathbb{N},\ n = kp.$$

Definition 13. An integer n is even if 2 is a divisor of n. Otherwise, the integer is said to be odd.

Remark 5. Combining the two definitions, we obtain that the set of even numbers is

$$\begin{aligned}\{n, \text{ such that } n \in \mathbb{N} \text{ and } 2 \text{ is dividing } n\} &= \{n, \text{ s.t. there exists } k \in \mathbb{N} \text{ and } n = 2k\} \\ &= \{2k, k \in \mathbb{N}\}.\end{aligned}$$

We will now look at two examples and give *direct proof* of the results.

Example 9. (Universal statement) For every number n divisible by 3, 9 is a divisor of n^2.

Example 10. (Conditional statement) Let n be an integer. If n is divisible by 3, then 9 is a divisor of n^2.

Since it is a conditional statement, the proof goes as follows: assume the hypothesis, deduce consequences logically until conclusion.

Proof. Let n be a number divisible by 3. We want to show that 9 is dividing n^2. By assumption, there exists $k \in \mathbb{N}$ such that

$$n = 3k.$$

It follows that $n^2 = (3k)^2 = 9k^2$. Since k^2 is an integer, 9 is dividing n^2. □

Example 11. (Existential statement) There exists a number divisible by 9 and 10.

Since it is an existential statement, the direct proof goes as follows: introduce a candidate, check that this candidate is indeed satisfying the conclusion.

Proof. Let us consider $x = 900$. We know that $x = 9 * 100$ so 9 is indeed a divisor of x. On the other hand $x = 90 * 10$, hence 10 is also a divisor of x.

Therefore there exists a number divisible by 9 and 10. □

Example 12. (Equivalence statement) Let n be an integer. n is divisible by 3 if and only if $n + 6$ is also divisible by 3.

There are two different ways to write a direct proof. The first one is to decompose the statement into two implications and prove each of them.

The second one is to assume one of the two assumptions and follow a chain of known equivalence. We will see such a proof in the next chapter.

2.2 More involved direct proofs

We now describe two reasonings that are often used in proofs. They are based on the following logic tautologies.

Theorem 2. Let p, q and r be three propositions, then

(i) Disjunction of case : $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \Rightarrow r$

(ii) Transitivity: $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$

Example 13. (Disjunction) Let a and b two real numbers. If $ab = 0$ then $a = 0$ or $b = 0$.

Proof. Let a and b be two real numbers such that $ab = 0$. Let us consider two cases:

- if $a = 0$ then the conclusion is true $a = 0$ or $b = 0$ and the implication is true.
- if $a \neq 0$ then $c = \frac{1}{a}$ is well defined and we can multiply the equality $ab = 0$ on both sides by c. We obtain

$$\begin{aligned} c(ab) &= c0 = 0 \\ \frac{1}{a}(ab) &= 0 \\ b &= 0 \end{aligned}$$

  so the conclusion is true.

By disjunction of cases, we proved the result. □

Example 14. (Transitivity) For every integer n, if n is divisible by 3 then $(n+6)^2$ is divisible by 9.

Proof. We proved previously that

- for every integer n divisible by 3, n^2 is divisible by 9.
- for every integer n divisible by 3, $n + 6$ is divisible by 3.

Let n be an integer divisible by 3, then by the second statement (left to right implication), we know that $n + 6$ is also divisible by 3.

Applying now the first statement to $m = n + 6$, we know that $m^2 = (n + 6)^2$ is divisible by 9. It concludes the proof. □

2.3 Indirect proof

We now present two methods that are called indirect and that are linked to the two following logic formulae.

Theorem 3. Let p, q and r be three propositions, then

(i) Contrapositive: $(p \to q) \Leftrightarrow (\neg q \to \neg p)$,

(ii) Contradiction: $(p \to q) \wedge \neg q \Rightarrow \neg p$,

Remark 6. "$\neg q \to \neg p$" is called the contrapositive of "$p \to q$"

Proof. Let us prove the contrapositive formula. It suffices to prove that for all propositions p and q, the proposition $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is true... use a truth table!!

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \to (\neg q \to \neg p)$	$(\neg q \to \neg p) \to (p \to q)$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T	T
F	T	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T

The proposition $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is always true, hence $(p \to q) \Leftrightarrow (\neg q \to \neg p)$. □

Exercice : Find the contrapositive of the proposition

"If there is a problem then there is a solution".

Solution :

"If there is no solution then there is no problem" (Shadock's motto).

Example 15. (Contrapositive) Let n be an integer. If n^2 is odd then n is odd.

The proof by contrapositive goes as follows: assume that the conclusion is false and prove that the assumption is false.

Proof. Let us describe first what would happen with a direct proof: let $n \in \mathbb{N}$. Assume that there exists $k \in \mathbb{N}$ such that

$$n^2 = 2k + 1,$$

and we are stuck since we can not exploit the fact that n^2 is a square.

Let us prove this result by the contrapositive. The contrapositive of the formula is "if n is not odd then n^2 is not odd" or equivalently " if n is even then n^2 is even".

Let n be an even number. There exists $k \in \mathbb{N}$ such that $n = 2k$. Hence $n^2 = 4k^2 = 2 * (2k^2)$ and n^2 is indeed even. □

Example 16. (Contradiction) Consider a right-angle triangle. We denote by c the length of the hypotenuse and by a and b the length of the two other sides. Then $c \leq a + b$.

The proof by contradiction goes as follows: assume that the statement is false and deduce a contradiction (that some other statement is both false and true).

Proof. We assume that

$$a + b < c.$$

Then by taking the square, we obtain that

$$a^2 + 2ab + b^2 < c^2$$

By Pythagoras' theorem, we know that $a^2 + b^2 = c^2$, hence $0 \leq 2ab < 0$. This is impossible, hence the contradiction. □

2.4 Induction

We denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of natural numbers. Elementary facts about the ordered set of natural numbers:

- (F1) Every $n \in \mathbb{N}$ has a successor $n+1$ s.t.
$$\forall m \in \mathbb{N}, n < m \Leftrightarrow n+1 \leq m.$$
- (F2) 0 is the least number.
- (F3) For all $m \in \mathbb{N} \setminus \{0\}$, there exists $n \in \mathbb{N}$, such that $m = n+1$.

Remark 7. There are different ways to construct the natural numbers:

- define real numbers and consider $\mathbb{N}$ as a subset of $\mathbb{R}$ and check that it satisfies the previous facts.
- define $\mathbb{N}$ by the previous facts and then define operations (addition, multiplication) on $\mathbb{N}$.

Weak principle of induction. Let $p(n)$ be a predicate on $\mathbb{N}$. Suppose the two following propositions hold

(i) $p(0)$ is true (basis)

(ii) $\forall\ n \in \mathbb{N},\ p(n) \Rightarrow\ p(n+1)$ (induction step).

Then for all $n \in \mathbb{N}$, $p(n)$ is true.

Remark 8. The weak principle of induction becomes a theorem if one admits the following property of $\mathbb{N}$: For all $A \subseteq \mathbb{N}$, if $0 \in A$ and for all $n \in A$, then $n+1 \in A$ then $A = \mathbb{N}$.
Indeed, let $p(n)$ be a predicate on $\mathbb{N}$ such that $p(0)$ is true and $\forall\ n \in \mathbb{N},\ p(n) \Rightarrow\ p(n+1)$. Let
$$A = \{n \in \mathbb{N}, p(n) \text{ is true}\}.$$
By assumption $0 \in A$ and if $n \in A$ then $n+1 \in A$. Therefore $A = \mathbb{N}$.

Example 17. Prove that $p(n) : 0 + 1 + \cdots + n = \sum_{t=0}^{n} t = \frac{n(n+1)}{2}$.

- Basis: $p(0)$ is true since $0 = 0$,
- Induction step: let $n \in \mathbb{N}$ such that $p(n)$ is true. Then
$$\begin{aligned} 0 + 1 + \cdots + (n+1) = \sum_{t=1}^{n+1} t &= \frac{n(n+1)}{2} + (n+1), \text{ by the induction assumption} \\ &= (n+1)\left(\frac{n}{2} + 1\right), \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$
Therefore, $p(n+1)$ is true.

- By the weak principle of induction, the formula is true for every $n \in \mathbb{N}$.

To be correct, the weak induction principle must be used with some precaution:

(i) The initialization step (basis) may start at some number $k \neq 0$.

Example 18. Prove that $n^2 \geq 3n$ for all $n \geq 3$.

- Basis: $p(3)$ is true since $9 \geq 9$,
- Induction step: let $n \geq 1$ such that $p(n)$ is true. Then

$$(n+1)^2 = n^2 + 2n + 1 \geq 5n + 1 = 3n + 2n + 1 \geq 3n + 3,$$

hence $p(n+1)$ is true.

- By the weak principle of induction, the formula is true for every $n \geq 3$. Observe that it is false for $n = 2$.

(ii) Be careful not to overlook the basis!

Example 19. Consider the property $p(n)$: n is greater than itself ($n > n$). From $p(n)$ it is easy to deduce $p(n+1)$: $n > n$ implies $n + 1 > n + 1$. However, $p(0)$ fails.

(iii) If the basis is $p(n_0)$, then do not forget to check that $p(n) \Rightarrow p(n+1)$ is shown for *every* $n \geq n_0$. The next example is Polya's proof that there is no horse of a different color. Find the mistake in the proof!

Example 20. For every $n \geq 1$, in a set of n horses, all horses have the same color.

Polya's proof:

- Basis $p(1)$: if there is only one horse, there is only one color.
- Induction step: assume that for any set of n horses, there is only one color. Take a set of $n+1$ horses, numbered $1, 2, \ldots, n+1$. Consider the sets $\{1, \ldots, n\}$, $\{2, \ldots, n+1\}$. Each set is of one color and since they overlap, $\{1, \ldots, n+1\}$ is of one color. Hence $p(n+1)$ holds.

The mistake is that the proof that $p(n) \Rightarrow p(n+1)$ is valid only for $n > 1$. Hence the step $p(1) \Rightarrow p(2)$ is missing.

Strong principle of induction. Let $p(n)$ be a predicate on $\mathbb{N}$. Suppose the two following propositions hold

(i) $p(0)$ is true (basis)

(ii) for all $n \in \mathbb{N}$, $(\forall m \leq n, p(m)) \Rightarrow p(n+1)$ (induction step).

Then for all $n \in \mathbb{N}$, $p(n)$ is true.

Example 21. Prove that any number $n \geq 2$ has a prime divisor.

- basis: $p(2)$ is true.
- Let $n \geq 2$ and assume that $p(2), \ldots, p(n)$ are true. Then, either $n+1$ is prime, in which case it has a prime divisor (itself). Or it is not, then there exist $2 \leq d < n+1$ which divides $n+1$. By induction hypothesis, d has a prime divisor, which is also a prime divisor of $n+1$.

Theorem 4. The strong principle of induction follows from the weak principle of induction.

Proof. Consider $p(n)$ and assume that (i) and (ii) hold in the strong induction principle. We denote by $q(n)$ the predicate $(\forall m \in \mathbb{N} \text{ s.t. } m \leq n,\ p(m))$. Let us show by using the weak induction principle that for every $n \in \mathbb{N}$ $q(n)$ is true.

- Basis: $q(0) = p(0)$, and therefore is true by (i).
- Induction step: let $n \in \mathbb{N}$, such that $q(n)$ is true. Then
  - for every $m \leq n$, $p(m)$ is true (by definition of $q(n)$)
  - $p(n+1)$ is true (by (ii)).

  Therefore $q(n+1)$ is true
- By the weak principle of induction, for every $n \in \mathbb{N}$, $q(n)$ is true, hence $p(n)$ is true.

□

Least Number Principle (infinite descent of Fermat). The LNP states: If $M \subseteq \mathbb{N}$ and $M \neq \emptyset$, then M has a least element.

Theorem 5. The Least Number Principle follows from the strong principle of induction.

Proof. We prove the result by contradiction. Assume that $M \subseteq \mathbb{N}$, $M \neq \emptyset$ and M has no least element.
Let $p(n)$ be the predicate defined on $\mathbb{N}$ by $p(n)$: "$n \notin M$". We prove by the strong induction principle that $p(n)$ is true for all $n \in N$, which contradicts the assumption $M \neq \emptyset$.

- Basis: $p(0)$ holds, otherwise 0 would be the least element.
- Induction step: assume that $p(m)$ holds for every $m \leq n$. This means that $m \notin M$ for every $m \leq n$. Then $n < m$ for all $m \in M$, which by Fact F1 is equivalent to $n+1 \leq m$ for all $m \in M$. It follows that $n+1 \notin M$, otherwise it would be the least element of M, a contradiction. Hence, $p(n+1)$ holds.

□

Theorem 6. The weak induction principle follows from the LNP.

Proof. Let $p(n)$ be a property such that $p(0)$ is true and $\forall n \in \mathbb{N}$, $p(n) \Rightarrow p(n+1)$. We prove that $\forall n \in \mathbb{N}$, $p(n)$, which amounts to showing that the class

$$M := \{n \in \mathbb{N} \text{ s.t } p(n) \text{ does not hold}\}$$

is empty. By the LNP, it is enough to show that M has no least element.
Suppose M has a least element, say m. Since $p(0)$ holds, $0 \notin M$, hence $m \neq 0$. Therefore, by (F3), there exists $n \in \mathbb{N}$ s.t. $m = n + 1$, hence $n < m$. Since m is a least element of M, $n \notin M$, i.e., $p(n)$ holds.
From our assumption, we have then $p(n + 1)$ holds, i.e., $p(m)$ holds, which means that $m \notin M$, a contradiction. □

As a consequence of Theorems 4, 5 and 6, all three principles are equivalent.

Chapter

3 Basic set theory

3.1 Basic definitions

Definition 14 (Naive definition of a set; this is Def. 6)**.** A set is a well-defined collection of objects. The objects in a set are called its elements. If A is a set,

(i) $a \in A$ means that a belongs to the set A, or that a is an element of A. Equivalently, we write $A \ni a$ (A contains a).

(ii) $a \notin A$ means that a does not belong to A or that a is not an element of A. Equivalently, we write $A \not\ni a$ (A does not contain a).

Example 22. Example of sets

- $A = \{1, 2, 4, 6\}$,
- $B = [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$,
- $C = \{1, 2, 4, 2, 6, 2\} = \{2, 1, 4, 6\}$.
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.
- $D = \{\{1, 2\}, 3, a, \{a, b\}\}$

Definition 15. A set A is a subset of a set B if every element of A is an element of B; one writes $A \subseteq B$ (A is included in B). Equivalently, we write $B \supseteq A$ (B includes A). Formally:

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \rightarrow x \in B).$$

Example 23.

- $\{1, 2\} \subset \{1, 2, Marc\}$,
- $\{1, 2, John\}$ is not a subset of $\{1, 2\}$.

Definition 16. Two sets A and B are equal, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Two sets which are equal contain the same elements. Whenever two sets are not equal we denote it by $A \neq B$. If $A \subseteq B$ and $A \neq B$, we have *strict inclusion*, denoted by $A \subset B$.[1] In order to prove that two sets A and B are equal, we have two methods:

[1] Many textbooks write $\subset$ for inclusion and $\subsetneq$ for strict inclusion.

(i) Prove that $A \subseteq B$ and $B \subseteq A$: fix $a \in A$ and show that it is in B, then fix $b \in B$ and show that it is in A.

(ii) Fix an element x and show that $x \in A$ if and only if $x \in B$.

A set can be defined via a property (this is called *specification*). Let A be a set and P be a predicate on A. There exists a set denoted $\{x \in A, P(x)\}$ or $\{x \in A \mid P(x)\}$ or $\{x \in A : P(x)\}$ or $\{x \in A \text{ s.t. } P(x)\}$ whose elements are the elements of $x \in A$ such that $P(x)$ is true.

Example 24. Let X be the set of animals.

- $P(x)$: "the animal x has feathers", $\{x \in X, P(x)\}$ is the set of animals with feathers.
- $Q(x)$: "the animal x has 3 legs", $\{x \in X, P(x)\}$ is the set of animals with 3 legs.

By taking the predicate $P(x) : x \neq x$, we obtain the *empty set*, denoted by $\emptyset$:

$$\emptyset = \{x \in A, x \neq x\}.$$

This set contains no element, since the predicate is always false.

Theorem 7. If X is a set, then $\emptyset \subseteq X$.

Proof. Recall that $\emptyset \subseteq X$ is the proposition

$$\forall x, (x \in \emptyset \to x \in X).$$

Therefore its negation is $\exists x, x \in \emptyset \wedge x \notin X$. This proposition is false since the empty set has no elements. Hence the initial proposition is true. □

Remark 9. If the above naive approach to set theory is sufficient most of the time, it cannot be used as a basis for the foundations of mathematics, as many paradoxes can be built with this loose definition of a set. The most famous one was proposed by Bertrand Russel: consider the set defined by the property that it contains only sets which do not belong to themselves:

$$S = \{x \mid x \text{ is a set such that } x \notin x\}$$

Then if $S \notin S$, it should belong to S (i.e., $S \in S$), and if $S \in S$, then it should not belong to S (i.e., $S \notin S$). This is why a rigorous axiomatic approach to set theory was constructed by Zermelo, and completed by Fraenkel, referred usually as the Zermelo-Fraenkel (ZF) axiomatic set theory.

3.2 Constructing new sets from operation on sets

We will now list different operations that can be used on sets and define new sets.

Definition 17. Let A and B be two sets. One can define

(i) (Pairing)[2] The set $\{A, B\}$ is the set with 2 elements A and B:

$$x \in \{A, B\} \text{ if and only if } (x = A \text{ or } x = B).$$

(ii) (Union) The set $A \cup B$ is the set containing both the elements of A and the elements of B:

$$x \in A \cup B \text{ if and only if } (x \in A \text{ or } x \in B).$$

(iii) (Intersection) The set $A \cap B$ is the set containing the elements that are both in A and in B:

$$x \in A \cap B \text{ if and only if } (x \in A \text{ and } x \in B).$$

(iv) (Relative complement) The set $A \setminus B$ (or $A - B$) is the set containing the elements that are in A and not in B:

$$x \in A \setminus B \text{ if and only if } (x \in A \text{ and } x \notin B).$$

(v) (Complement) Suppose U is some universal set. If $A \subseteq U$,

$$A^c := U \setminus A.$$

(vi) (Symmetric difference) The set $A \Delta B$ is the set containing the elements in $A \cup B$ and not in $A \cap B$.

$$A \Delta B := (A \cup B) \setminus (A \cap B).$$

Example 25. Union, intersection and relative complement are quite straightforward. One must be careful with the pairing, especially when pairing a set with itself: a set A and the singleton $\{A\} = \{A, A\}$ are two different sets. For example,

- If $A = \emptyset$, the set $\emptyset$ does not contain any element, while the set $\{\emptyset\}$ contains exactly one element: the element $\emptyset$!
- $A = \emptyset$, $B = \{\emptyset\}$, $C = \{B\} = \{\{\emptyset\}\}$, are three different sets.
- $A = \{1, 2\}$, $B = \{2, 3\}$ and $\{A, B\} = \{\{1, 2\}, \{2, 3\}\}$.

With the previous propositions and the results that we have shown in the chapter on logic, we can show the following rules of computation. They are theorems, hence have to be "proved" from definitions and previous results.

Theorem 8. The union $\cup$ satisfies the following properties:

(i) $A \cup \emptyset = A$;

(ii) $A \cup B = B \cup A$;

(iii) $(A \cup B) \cup C = A \cup (B \cup C)$.

[2] In ZF, pairing is an axiom permitting to construct a new set from two sets.

Theorem 9. The intersection $\cap$ satisfies the following properties:

(i) $A \cap \emptyset = \emptyset$;

(ii) $A \cap B = B \cap A$;

(iii) $(A \cap B) \cap C = A \cap (B \cap C)$.

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem 10 (De Morgan's laws)**.** Let A, B and C be three sets:

(i) $A \setminus \emptyset = A$

(ii) $\emptyset \setminus A = \emptyset$

(iii) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$ in particular $(A \cup B)^c = A^c \cap B^c$.

(iv) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$ in particular $(A \cap B)^c = A^c \cup B^c$

Proof. As exercice. □

Theorem 11. The symmetric difference Δ satisfies the following properties:

(i) $A\Delta\emptyset = A$;

(ii) $A\Delta B = B\Delta A$;

(iii) $(A\Delta B)\Delta C = A\Delta(B\Delta C)$.

(iv) $A\Delta A = \emptyset$

3.3 Family of sets

We have already seen with the pairing axiom that one can consider sets whose elements are also sets. These are "sets of sets", usually called "collections of sets" or "family of sets".

Example 26. Let $A = \{a, b\}$. We have:

- $\{\{a, b\}\}$ is a set with one element which is A.
- $\{\{a\}, \{a, b\}\}$ is a set with two elements.
- $\{\emptyset, \{a\}, \{b\}\}$ is a set with three elements.

In particular the following set, called the power set, is very important

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

It is the set of all subsets of A.

Definition 18. Given any set A, the set denoted by 2^A (or $\mathcal{P}(A)$) and called *power set of* A is defined[3] by:

$$B \text{ is a member of } 2^A \text{ if and only if } B \subseteq A.$$

Formally:

$$\mathcal{P}(A) = \{B, B \subseteq A\}$$

Note that we have always $\emptyset \in 2^A$ and $A \in 2^A$.

Example 27.

- $2^\emptyset = \{\emptyset\}$.
- $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
- $2^{\{a,b,c\}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
- $\mathcal{P}(\mathbb{R})$ is the set of all subsets of $\mathbb{R}$.

We can now extend the notion of union and intersection to family of sets.

Definition 19. Let $\mathcal{F}$ be a family of sets. The union and intersection of all sets in the family is defined as follows:

$$\bigcup \mathcal{F} = \{x,\ \exists A(A \in \mathcal{F} \wedge x \in A)\}$$

and

$$\bigcap \mathcal{F} = \{x,\ \forall A(A \in \mathcal{F} \to x \in A)\}$$

Definition 20. Let $\mathcal{F}$ be a family of sets such that $\mathcal{F} = \{A_i, i \in I\}$. The union and intersection of all sets in the family is defined as follows:

$$\bigcup_{i \in I} A_i = \{x,\ \exists i(i \in I \wedge x \in A_i)\}$$

and

$$\bigcap_{i \in I} A_i = \{x,\ \forall i(i \in I \to x \in A_i)\}$$

Remark 10. When $\mathcal{F}$ can be indexed by some set I, both definitions are available and are equivalent (union and intersection).

With these two new definitions, we can extend De Morgan's Law and the distributivity formulas.

Theorem 12. Let $(A_i)_{i \in I}$ a family of sets and B a set:

- $(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c$,
- $(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$,
- $B \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (B \cap A_i)$,
- $B \cup (\bigcap_{i \in I} A_i) = \bigcap_{i \in I} (B \cup A_i)$.

[3] In ZF, this is an axiom saying that if A is a set, then 2^A is also a set.

3.4 Cartesian product

Definition 21 (Naive definition of an ordered pair)**.** An ordered pair (x, y), or simply a pair, is a list of two objects x and y given in a definite order; x is said to be the first (or left) coordinate of the pair, while y is the second (or right) coordinate.

A formal definition due to Kuratowski and using only sets is as follows: An ordered pair (x, y) is the set which contains the singleton $\{x\}$, and the pair $\{x, y\}$:

$$(x, y) = \{\{x\}, \{x, y\}\} \in 2^{2^{A\cup B}}.$$

As pairs are ordered, equality of pairs means equality of their components "coordinatewise":

$$(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d).$$

In the Kuratowski's definition, this statement has to be proved[4]

Definition 22. The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$, that is:

$$A \times B = \{x, (\exists a \in A)(\exists b \in B), x = (a, b)\}.$$

- If $A = B$ we simply denote $A \times A$ by A^2.
- The Cartesian product of three sets A, B and C, denoted $A\times B\times C$, is the abbreviation of the set $(A \times B) \times C$. The elements of $A \times B \times C$ are called the ordered triplets of A, B and C.

[4]Here is the proof:

- " $\Leftarrow$ " If $a = c$ and $b = d$, then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Thus $(a, b) = (c, d)$.
- " $\Rightarrow$ ": Two cases:

  (i) If $a = b$:

  $$(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\},$$
  $$(c, d) = \{\{c\}, \{c, d\}\} = \{\{a\}\}.$$

  Thus $\{c\} = \{c, d\} = \{a\}$, which implies $a = c$ and $a = d$. By hypothesis, $a = b$. Hence $b = d$.

  (ii) If $a \neq b$, then $(a, b) = (c, d)$ implies

  $$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

  – Suppose $\{a\} = \{c, d\}$. Then $c = d = a$, and so

  $$\{\{c\}, \{c, d\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}.$$

  But then $\{\{a\}, \{a, b\}\}$ would also equal $\{\{a\}\}$, so that $b = a$ which contradicts $a \neq b$.

  – Suppose $\{a\} = \{c\}$. Then $a = c$ and we have

  $$\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, d\}\}$$

  Since $a \neq b$, we have $\{a, b\} = \{a, d\}$ and therefore $b = d$.

Proposition 2. (i) $A \times B = \emptyset$ is equivalent to $(A = \emptyset \vee B = \emptyset)$.

(ii) If $A \neq \emptyset$ and $B \neq \emptyset$ then

(a) $A \times B \subseteq A' \times B'$ is equivalent to $(A \subseteq A' \wedge B \subseteq B')$.

(b) $(A \times B) \cup (A' \times B) = (A \cup A') \times B$.

(c) $(A \times B) \cap (A' \times B) = (A \cap A') \times B$.

Proof. (i) $\neg(A = \emptyset \vee B = \emptyset) \Leftrightarrow (A \neq \emptyset \wedge B \neq \emptyset) \Leftrightarrow ((\exists a \in A) \wedge (\exists b \in B)) \Leftrightarrow ((a, b) \in A \times B) \Leftrightarrow \neg(A \times B = \emptyset)$

(ii) Let $A \neq \emptyset$ and $B \neq \emptyset$:

(a) "$\Rightarrow$": Suppose $A \times B \subseteq A' \times B'$. Let $a \in A$ and $b \in B$. By hypothesis $(a, b) \in A \times B$ implies $(a, b) \in A' \times B'$ which implies by definition of the cartesian product that: $(a \in A') \wedge (b \in B')$.
"$\Leftarrow$": Suppose $(A \subseteq A') \wedge (B \subseteq B')$. Let $x \in A \times B$, then $(\exists a \in A) \wedge (\exists b \in B)$ such that $x = (a, b)$, but $(A \subseteq A') \wedge (B \subseteq B')$ implies $(a \in A') \wedge (b \in B')$. Hence $x = (a, b) \in A' \times B'$.

□

Chapter 4

Functions

4.1 Definition of a function

Definition 23. • A mapping from the set A to the set B, denoted by $f : A \to B$, is a triplet $(A, B, graph(f))$ where A and B are two sets and $graph(f)$ is a subset of $A \times B$ such that for every $a \in A$, there is one and only one $b \in B$ such that $(a, b) \in graph(f)$:

$$[\forall x \in A, \exists y \in B, (x, y) \in graph(f)] \quad \wedge \quad [((a, b) \in graph(f) \wedge (a, c) \in graph(f)) \Rightarrow (b = c)].$$

In other words, for every $a \in A$ there is exactly one element denoted $f(a) \in B$ such that the ordered pair $(a, f(a)) \in graph(f)$.

- The set A is called the domain of f;
- The set B is called the codomain of f;
- The set $graph(f)$ is called the graph of f;
- The unique element $f(a)$ such that $(a, f(a)) \in graph(f)$ is called the image of a by f;
- If $C \subseteq A$, the set $f(C) := \{b \in B, \exists a \in C, (a, b) \in graph(f)\} = \{b \in B, \exists a \in C, b = f(a)\}$ is called the image of C by f.
- The set $f(A) := \{b \in B, (a, b) \in graph(f)\} = \{b \in B, \exists a \in A, b = f(a)\}$ [1] is called the image of f.

• We denote by B^A (or $\mathcal{F}(A, B)$) the set of functions from A to B.

Observe that:

$$graph(f) = \{(a, b) \in A \times B, b = f(a)\}.$$

Note that

- $C = \emptyset$ is equivalent to $f(C) = \emptyset$.
- If $f : A \to B$, $f(\{x\}) = \{f(x)\}$ for every $x \in A$.

[1] Observe that $f(A) \subseteq B$

Definition 24. Two mappings f and g from A to B are said to be equal if for every element $a \in A$ one has $f(a) = g(a)$.

Example 28. Here are some examples of mappings:

- The identity mapping on A, denoted by id_A, is the mapping from A to A defined by $id_A(a) = a$ for every $a \in A$.
- A mapping $f : A \to B$ is said to be constant if for every a and a' in A, one has

  $$f(a) = f(a').$$

  In other words, there exist an element $b \in B$ such that for every $a \in A$, $f(a) = b$.
- The mapping $proj_1 : A \times B \to A$ (resp. $proj_2 : A \times B \to B$) which associates to the pair (a, b) the element a (resp. b) is called the canonical projection of $A \times B$ on A (resp., B).
- Let $C \subseteq A$. The restriction of the mapping $f : A \to B$ to C is the mapping $f_{|C} : C \to B$ defined by $f_{|C}(x) := f(x)$ for every $x \in C$.
- Let $B \subseteq A$. The characteristic function of the set B is the function $\chi_B : A \longrightarrow \{0, 1\}$ defined by

  $$\chi_B(a) = \begin{cases} 1, & \text{if } a \in B \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3. Let $f : A \to B$ and let A_1, A_2 be two subsets of A. Then

(i) $A_1 \subseteq A_2$ implies $f(A_1) \subseteq f(A_2)$,

(ii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,

(iii) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Note that the above inclusion may not be an equality. Find a counter-example.

Definition 25. The inverse image of $C \subseteq B$ by $f : A \to B$ is the set

$$f^{-1}(C) = \{a \in A, f(a) \in C\}.$$

Proposition 4. Let $f : A \to B$ and let B_1, B_2 be two subsets of B. Then

(i) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$,

(ii) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

4.2 Injection, Surjection, Bijection.

Given a function f from A to B that associate to a an element $f(a)$, one could want to define an inverse function from B to A that associates to $f(a)$ the element a. Unfortunately, two problems may arise when trying to define such a function

- an element of B may not be in relation with any element in A.
- an element of B may be in relation with several elements in A.

It is interesting to focus on functions where one or both of these problems do not occur. This yields the two following definitions.

Definition 26. A mapping $f : A \to B$ is said to be surjective (or onto) if every point in B is the image of a point in A that is

- $B = f(A)$,

  or equivalently

- for all $b \in B$, there exists $a \in A$, such that $f(a) = b$.

Informally, if f is surjective every elements of B has at least one "pre-image" by f.

Example 29.

- $f : \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = x^2$ is surjective,
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective.

Definition 27. A mapping $f : A \to B$ is said to be injective (or one- to-one) if two distinct elements of A have different images by f. That is,

- if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$,

  or equivalently

- if $f(a_1) = f(a_2)$ then $a_1 = a_2$ (By using the contrapositive).

Informally, if f is injective, every element of B has at most one "pre-image" by f.

Example 30.

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not injective,
- $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = x^2$ is injective.

Avoid the confusion between the definition of injectivity and the fact that every mapping as the property that $a_1 = a_2$ implies that $f(a_1) = f(a_2)$, which simply means that every element has a unique image.

Definition 28. A mapping $f : A \to B$ is said to be bijective if it is both surjective and injective.

Definition 29. Let $f : A \to B$ and $g : B \to C$. The composition of f by g is the mapping $g \circ f : A \to C$ defined by

$$g \circ f(a) = g(f(a)), \qquad \forall a \in A.$$

Proposition 5. Given $f : A \to B$, $g : B \to C$ and $h : C \to D$ then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Indeed let $a \in A$,

$$h \circ (g \circ f)(a) = h((g \circ f)(a)) = h(g(f(a))),$$

whereas

$$(h \circ g) \circ f(a) = (h \circ g)(f(a)) = h(g(f(a)).$$

Definition 30. Given a set A, we denote by id_A and called identity mapping, the mapping defined by

$$\forall a \in A id_A(a) = a$$

Definition 31. A mapping $f : A \to B$ is invertible if there exists a mapping $g : B \to A$ such that:

- $g \circ f = id_A$,
- $f \circ g = id_B$.

When the mapping g exists, it is unique. One denote it by f^{-1}, and called it the inverse mapping of f.

Proof. (Proof of unicity) Let us assume that f is invertible and there exists a least two mappings satisfying the assumptions:

- Let $g_1 : B \to A$ such that $g_1 \circ f = id_A$ and $f \circ g_1 = id_B$.
- Let $g_2 : B \to A$ such that $g_2 \circ f = id_A$ and $f \circ g_2 = id_B$.

We now prove that $g_1 = g_2$. Since $g_1 \circ f = g_2 \circ f$ we have, by composition by f,

$$(g_1 \circ f) \circ g_1 = (g_2 \circ f) \circ g_1.$$

Hence

$$g_1 \circ (f \circ g_1) = g_2 \circ (f \circ g_1),$$

which implies

$$g_1 \circ (id_B) = g_2 \circ (id_B).$$

Hence $g_1 = g_2$. □

Note that the notation f^{-1} may have two different meanings:

- **if $C \subset B$, $f^{-1}(C)$ is the inverse image of a set C and always defined.**
- **if $y \in B$, then $f^{-1}(y)$ is the image of $y \in B$ by the inverse mapping and only defined if f is invertible.**

Proposition 6. Let $f : A \to B$ be a function. Then f is injective if and only if for every $A_1, A_2 \subset A$, we have

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2).$$

Proof.

- " $\Rightarrow$ " the inclusion $\subseteq$ is always true. We show the inclusion $\supseteq$: let $y \in f(A_1) \cap f(A_2)$ and let us show that $y \in f(A_1 \cap A_2)$.

  We have in particular that $y \in f(A_1)$, hence there exists $x_1 \in A_1$ such that $f(x_1) = y$. Similarly, we have $y \in f(A_2)$, hence there exists $x_2 \in A_2$ such that $f(x_2) = y$.

  Since f is injective and $f(x_1) = f(x_2)$, we obtain $x_1 = x_2 =: x$ and x is both in A_1 and in A_2 therefore $x \in A_1 \cap A_2$. Hence there exists $x \in A \cap B$ such that $f(x) = y$: $y \in f(A \cap B)$.

- " $\Leftarrow$ " Let $x \neq y$. we have immediatly

  $$f(\{x\} \cap \{y\}) = f(\{x\}) \cap f(\{y\}) = \emptyset,$$

  Which implies $f(x) \neq f(y)$.

□

Proposition 7. Let $f : A \to B$ and $g : B \to C$.

- f and g injective $\Rightarrow$ $g \circ f$ injective
- f and g surjective $\Rightarrow$ $g \circ f$ surjective
- $g \circ f$ injective $\Rightarrow$ f injective
- $g \circ f$ surjective $\Rightarrow$ g surjective

Proof. As exercice. □

Proposition 8. A mapping $f : A \to B$ is bijective if and only if it is invertible.

Proof.

- "$\Rightarrow$": For every $b \in B$, we consider the set

  $$f^{-1}(\{b\}) = \{a \in A, \text{ such that } f(a) = b\}.$$

  Since f is surjective, we know that this set is non-empty. Since f is injective it is a singleton. We denote by $g(b)$ its unique element.

By construction, for every $b \in B$, $g(b) \in f^{-1}(\{b\})$ so $f(g(b)) = b$.

Moreover, for every $a \in A$, $f(a) \in \{f(a)\}$ so $a \in f^{-1}(\{f(a)\}) = \{g(f(a))\}$. Hence$g(f(a)) = a$.

- "$\Leftarrow$": Let $g : B \to A$ be a mapping which satisfies $f \circ g = id_B$ and $g \circ f = id_A$.
  - id_A is bijective therefore it is injective. By Proposition 7, $g \circ f$ injective implies f injective.
  - id_B is bijective therefore it is surjective. By Proposition 7, $f \circ g$ surjective implies f surjective. We conclude that f is bijective.

□

Chapter

5 Relations

5.1 Definition of a relation

Definition 32. A *binary relation* $\mathcal{R}$ on a set A is a subset of the Cartesian product $A^2 = A \times A$.

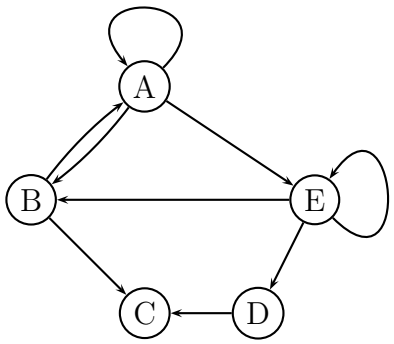
Example 31.

- Let E be a set. The diagonal $=_E$ (or $\Delta(E)$) of E is the subset of E^2 defined by
$$=_E := \{(x, x) \in E^2, x \in E\}$$
$=_E$ can be see as a relation, and if $(x, y) \in =_E$ then we have $x =_E y$ or more commonly $x = y$.
- If $f : A \mapsto B$, then $graph(f)$ is a relation between A and B.
- $<$ on the set of real numbers is a relation.
- If X is the set of human being, $x\mathcal{R}y$ if and only if x is the friend of y define a relation on X.

One can represent a relation as a graph or a table. For example : the relation $\mathcal{R}$ on $\{A, B, C, D, E\}$ defined by

$$\mathcal{R} := \{(A, A), (E, E), (A, B), (B, A), (A, E), (B, C), (D, C), (E, B), (E, D)\},$$

can be represented by

- 

-

	A	B	C	D	E
A	1	1	0	0	1
B	1	0	1	0	0
C	0	0	0	0	0
D	0	0	1	0	0
E	0	1	0	1	0

Definition 33. Given a binary relation $\mathcal{R}$, we define the *inverse relation* denoted by $\mathcal{R}^{-1}$ as follows

$$\mathcal{R}^{-1} = \{(y, x), (x, y) \in A^2, (x, y) \in \mathcal{R}\}$$

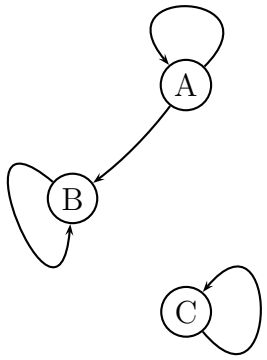
or equivalently $y\mathcal{R}^{-1}x$ if and only if $x\mathcal{R}y$.

Definition 34. A relation $\mathcal{R}$ on $E \times E$ is said to be:

- *reflexive* if:

$$\forall x \in E, x\mathcal{R}x$$

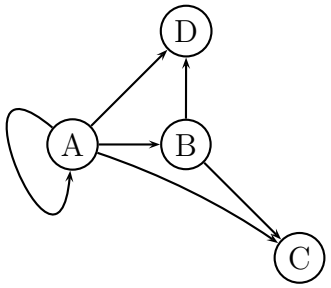
Example:



- *transitive* if:

$$\forall (x, y, z) \in E^3, (x\mathcal{R}y) \wedge (y\mathcal{R}z) \Rightarrow (x\mathcal{R}z)$$

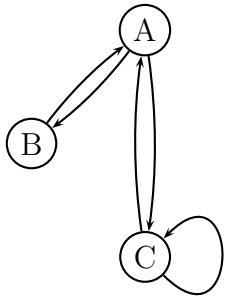
Example:



- *symmetric* if:

$$\forall (x, y) \in E^2, x\mathcal{R}y \Leftrightarrow y\mathcal{R}x.$$

Example:

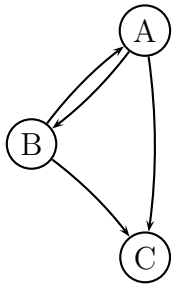


- *anti-symmetric* if
$$\forall (x, y) \in E^2, x\mathcal{R}y \wedge y\mathcal{R}x \Rightarrow x = y.$$

  Example: Let X be a set, one can see $\subseteq$ as an anti-symetric relation on 2^X.

- *total (or complete)* if
$$\forall (x, y) \in E^2, x\mathcal{R}y \vee y\mathcal{R}x \text{ is true.}$$

  Example:



5.2 Equivalence relation

Definition 35. An *equivalence relation* on E is a relation on $E \times E$ which is reflexive, symmetric, and transitive.

Example 32. Here are some standard examples of equivalence relations:

- The equality relation $=_E$.
- The equality between subsets of E, that is the equality $=_{2^E}$.
- Given a set E, the relation in $2^E \times 2^E$ defined by "there exists a bijective mapping $f : A \mapsto B$."

Definition 36. Let $\mathcal{R}$ be an equivalence relation on E and $x \in E$, the *equivalence class* of x for $\mathcal{R}$ is the subset of E:
$$\mathcal{R}(x) = \{y \in E | x\mathcal{R}y\}$$

Definition 37. We say that $\Pi \subseteq 2^E$ is a *partition* of a set E if it satisfies:

(i) Two distinct elements of Π are disjoint, that is:

$$\forall P \in \Pi,\ \forall Q \in \Pi,\ (P \neq Q)) \Rightarrow (P \cap Q = \emptyset).$$

(ii) Every element x of E belongs to some $P \in \Pi$, that is

$$\bigcup_{P \in \Pi} P = E$$

Remark 11. The first condition can be rewritten as

$$\forall P \in \Pi,\ \forall Q \in \Pi,\ (P \cap Q) \neq \emptyset) \Rightarrow (P = Q).$$

Remark 12. In order to prove that some $\Pi \subset 2^E$ is not a partition, one has to prove one of the two following results:

- there exists $x \in E$ such that $x \notin \cup_{P \in \Pi} P$
- there exists $P \in \Pi$ and $Q \in \Pi$ such that $P \neq Q$ and $P \cap Q \neq \emptyset$.

Example 33. Let $E = \{1, 2, 3, 4, 5\}$. Then

- $\Pi = \{\{1, 2\}, \{3, 4\}, \{5\}\}$ is a partition.
- $\Pi = \{\{1, 2\}, \{3, 4\}\}$ is not a partition.
- $\Pi = \{\{1, 2\}, \{3, 4, 5\}, \{5\}\}$ is not a partition.

Proposition 9. Let $x, y \in E$ such that $x\mathcal{R}y$ then $\mathcal{R}(x) = \mathcal{R}(y)$.

Proof. Let $x, y \in E$ such that $x\mathcal{R}y$. We show the equality by double inclusion. Let $z \in \mathcal{R}(x)$. By definition, we have

$$x\mathcal{R}z$$

Since $\mathcal{R}$ is symmetric, we know that $y\mathcal{R}x$. By transitivity, we deduce that $y\mathcal{R}z$ and thus $z \in \mathcal{R}(y)$. We prove similarly that if $z \in \mathcal{R}(y)$ the $z \in \mathcal{R}(x)$ and therefore the two sets are equal. □

Theorem 13. Let E be a set, the two following propositions are equivalent:

(i) Π is a partition of E.

(ii) There exists an equivalence relation $\mathcal{R}$ on E such that $\Pi = \{\mathcal{R}(x), x \in E\}$.

Where $\mathcal{R}(x)$ is the equivalence class of x for $\mathcal{R}$.

Proof. See Exercise. □

Definition 38. Let $\mathcal{R}$ be an equivalence relation on E, the *quotient set* of E by $\mathcal{R}$ is the set $E/\mathcal{R}$ of equivalence classes[1] of $\mathcal{R}$, that is the set:

$$E/\mathcal{R} := \{\mathcal{R}(x), x \in E\} \subseteq 2^E.$$

[1] It is a partition of E.

5.3 Order relation

Definition 39.
- A *preorder relation* on E is a relation reflexive and transitive.
- An *order relation* on E is a relation reflexive, transitive and antisymmetric;
- A *total (or complete) preorder relation* on E is a relation reflexive, transitive and total.

Example 34.
- If E is a set, $\subseteq$ is an order relation on 2^E but not a total preorder: assume $E = \{1, 2\}$ then $\{1\}$ and $\{2\}$ are not comparable.
- $=_E$ is the unique preorder on E which is an order relation and an equivalence relation.
- If $E = \mathbb{R}$ and $\leq$ induce a total order relation.

For the three following definition, we fix $\Subset$ an order relation on E, and X a subset of E.

Definition 40.
- $m \in E$ is a *lower bound* of X for $\Subset$ if for every element $x \in X$ one has $m \Subset x$. When X has a lower bound, it is said to be *bounded below*.
- $M \in E$ is an *upper bound* of X for $\Subset$ if for every element $x \in X$ one has $x \Subset M$. When X has an upper bound, it is said to be *bounded above*.

Definition 41.
- $m \in E$ is an *infimum* of X for $\Subset$ if:
  (i) m is a lower bound of X for $\Subset$.
  (ii) it is the greatest lower bound: for every $x \in E$ lower bound of X, one has $x \Subset m$.

  When it exists, it is unique and denoted by $\inf(X)$ or $\inf_{x \in X} x$.
- $M \in E$ is a *supremum* of X for $\Subset$ if:
  (i) M is an upper bound of X for $\Subset$.
  (ii) it is the smaller upper bound: for every $x \in E$ upper bound of E, one has $M \Subset x$.

  When it exists, it is unique and denoted by $\sup(X)$ or $\sup_{x \in X} x$.

Definition 42.
- $m \in E$ is a *minimum* of X for $\Subset$ if:
  (i) m is a lower bound of X.
  (ii) $m \in X$.

  When it exists, it is unique and denoted by $\min(X)$ or $\min_{x \in X} x$.
- $M \in E$ is a *maximum* of X for $\Subset$ if:

(i) M is a upper bound of X)

(ii) $M \in X$.

When it exists, it is unique and denoted by $\max(X)$ or $\max_{x \in X} x$.

Proof. (unicity) See Exercices. □

Example 35. Consider $E = \mathbb{R}$ and the order relation $\leq$ then. Let $X =]0, 1]$ then

- $\inf(X) = 0$
- X has no minimum,
- -1 is a lower bound of X,
- $\inf(X) = 1$,
- $\max(X) = 1$,
- 0.5 is not an upper bound of X.

Example 36. Let $E = \{1, 2, 3\}$. We consider the order relation $\subseteq$ on $2^E \setminus \{\emptyset\}$ (power set without the element empty set) and $X = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$.

- X has no lower bound,
- $\sup(X) = \{1, 2, 3\}$,
- X has no maximum,
- X has an infimum in 2^E.

Example 37. Let E be a set. We consider the order relation $\subseteq$ on 2^E. Let A and B be two subsets of E. We have:

$$\inf(\{A, B\}) = A \cap B$$
$$\sup(\{A, B\}) = A \cup B$$

Proposition 10. Let $\Subset$ be an order relation on E and X a subset of E.

$$\min(X) \text{ exists } \underset{\not\Leftarrow}{\Rightarrow} \inf(X) \text{ exists } \underset{\not\Leftarrow}{\Rightarrow} X \text{ has a lower bound.}$$

$$\max(X) \text{ exists } \underset{\not\Leftarrow}{\Rightarrow} \sup(X) \text{ exists } \underset{\not\Leftarrow}{\Rightarrow} X \text{ has an upper bound.}$$

Proof. See Exercises. □

Theorem 14. Every bounded set A of $(\mathbb{R}, \leq)$ has a supremum and an infimum.

5.4 Complements: Decomposition of a function

Theorem 15. Every mapping $f : E \to F$ can be written as the composition of an injection i, a bijection b and a surjection s:

$$f = i \circ b \circ s$$

The idea is to introduce two intermediate well chosen sets: the set $f(E)$ and the set $E/\mathcal{R}$ where the relation[2] $\mathcal{R}$ is defined by $x\mathcal{R}y$ if $f(x) = f(y)$. We consider then the following chain of sets

$$E \to E/\mathcal{R} \to f(E) \to F.$$

Proof. Let f be a function from E to F, we define $\mathcal{R}$ by

$$\forall (x, y) \in E^2, x\mathcal{R}y \text{ if } f(x) = f(y).$$

This is an equivalence relation.

- reflexive: let $x \in E$, then $f(x) = f(x)$ so $x\mathcal{R}x$.
- symmetric: if $x, y \in E$ are such that $x\mathcal{R}y$, it follows that $f(x) = f(y)$ and thus $y\mathcal{R}x$.
- transitive: if $x, y, z \in E$ are such that $x\mathcal{R}y$ and $y\mathcal{R}z$, it follows that $f(x) = f(y) = f(z)$ and $x\mathcal{R}z$.

We define now three mapping:

- $s : E \to E/\mathcal{R}$ which associates to each $x \in E$ the equivalence class $\mathcal{R}(x)$
- $b : E/\mathcal{R} \to f(E)$ which associates to each $\mathcal{R}(x) \in E/\mathcal{R}$ the unique element $f(x)$ of the singleton $f(\mathcal{R}(x))$
- $i : f(E) \to F$ which associates to each $y \in f(E)$ the element $y \in F$ (identity but with two different sets).

Let us check that this three mapping are satsifying the announced properties:

- s: By definition of $E/\mathcal{R}$ as the set of equivalence class, every element is the recurrence class of someone: s is surjective.
- i: Since the mapping is the identity, it is clear that for all $x, y \in f(E)$, $i(x) = i(y)$ implies $x = y$.
- b: Exercise

□

[2] See Chapter 5

Chapter

6 Cardinality

The aim of this chapter is to answer the question: How many elements does a set have? In fact, that will be more to compare the number of elements between two sets.

6.1 Definition

Let $p, n \in \mathbb{N}$, we denote by $\{p, \dots, n\}$ the set defined by

$$\{p, \dots, n\} := \{k \in \mathbb{N}, p \leq k \leq n\}.$$

Proposition 11. If $f : E \to \{1, \dots, n\}$ and $g : E \to \{1, \dots, p\}$ are two bijections, then $n = p$.

Proof.

(i) Suppose $p \geq n$. f is a bijection, then f^{-1} is a bijection, therefore $h_1 = g \circ f^{-1} : \{1, \dots, n\} \to \{1, \dots, p\}$ is a bijection. Hence h_1 is a surjection, therefore $\forall j \in \{1, \dots, p\}, \exists i \in \{1, \dots, n\}$ such that $h_1(i) = j$ which implies $n \geq p$. $\Rightarrow n = p$.

(ii) Suppose $n \geq p$, then applying the previous reasonning to $g^{-1} \circ f$ yields the same result.

□

Definition 43. A set E is finite if $E = \emptyset$ or if there exists $n \in \mathbb{N} \setminus \{0\}$ and a bijection $f : E \to \{1, \dots, n\}$. By the previous Proposition, such a n, if it exists, is unique. It is called the cardinality of E and denoted by $|E|$. We put by convention $|\emptyset| = 0$.

Proof. Assume that there exist two integers n and p satisfying the definition. Then, there exist two bijections $f : E \to \{1, \dots, n\}$ and $g : E \to \{1, \dots, p\}$ hence $n = p$. This proves the unicity of n. □

Definition 44. A set which is not finite is said to be infinite.

Definition 45. We say that two sets A and B have the same cardinality (or same power) if there is a bijection between A and B.

Remark 13. For two finite sets, the existence of a bijection between A and B implies that $|A| = |B|$, hence this is coherent: they have the same cardinality.

Definition 46. A set with the same cardinality than $\mathbb{N}$ is said to be countably finite. Sets being finite or countably infinite are countable. Others are uncountable.

6.2 Propositions

Theorem 16. (Cantor-Schröder-Bernstein) Let A and B be two sets. Exactly one of the three following cases can happen:

- There exists an injection from A to B but no injection from B to A. (In this case, there is a surjection from B to A but no surjection from A to B).
- There exists an injection from B to A, but no injection from A to B. (In this case, there is a surjection from A to B but no surjection from B to A).
- There exists an injection from A to B and from B to A. In this case there also exists surjection in both directions and even bijections.

Theorem 17. Let $n \in \mathbb{N}$, the union of n countable sets is countable.

Example 38. In particular, $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ is countable. There exists a bijection between $\mathbb{N}$ and $\mathbb{Z}$: for all $n \in \mathbb{N}$,

$$g(2n) = n,$$

and

$$g(2n+1) = -(n+1).$$

g is a bijection from $\mathbb{N}$ to $\mathbb{Z}$:

- It is surjective: let $z \in \mathbb{Z}$ then either $z \in \mathbb{N}$ and $g(2z) = z$ or $z \in (-\mathbb{N}) \setminus \{0\}$ and $g(2(-z)+1) = z$.
- It is injective: it is clear that g only take every strictly positive or negative value at most once. Be careful about 0.

Theorem 18. Let $n \in \mathbb{N}$, the cartesian product of n countable sets is countable.

Example 39. In particular, $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable. There exists a bijection between $\mathbb{N}^2$ and $\mathbb{N}$: drawing. The theorem yields another method to prove that $\mathbb{N}^2$ is countable: there exists an injection from $\mathbb{N}^2$ to $\mathbb{N}$

$$\forall p, q \in \mathbb{N}, g(p,q) = 2^p * 3^q,$$

and a surjection

$$\forall p, q \in \mathbb{N}, g(p,q)) = p + q.$$

By Theorem 16, the only possible case then it the third one: the two sets are in bijection.

Proposition 12. $\mathbb{Q} = \{\frac{p}{q}, p, q \in \mathbb{Z} \times (\mathbb{N} \setminus \{0\}\}$ is a countable set.

Proof. We use the previous theorem.

- First $\mathbb{N} \subset \mathbb{Q}$, hence there exists a canonical injection (the identity mapping).

- Second there exists a bijection between $\mathbb{N}$ and $H = \mathbb{Z} \times (\mathbb{Z} \setminus 0)$. And there exists a surjection from H to $\mathbb{Q}$:
$$\forall (p, q) \in H,\ f(p, q) = \frac{p}{q}.$$
By composition, we obtain a surjective mapping from $\mathbb{N}$ to $\mathbb{Q}$.

By Theorem 16, the two sets are in bijection. □

Proposition 13. $\mathbb{R}$ is not countable.