Probability and Statistics

December 7th, 2021, 9.30 - 11:30

Erasmus Mundus Joint Master Degree (EMJMD) - QEM1 (First Year) - 2021/2022

EX. 1), 2), 3) are mandatory. Then students have to choose between EX. 4a), 4b). No calculator and no notes allowed.

1) **(30 mins.)** Given X, Y bivariate continuous r.v. with density

$$f(x,y) = \begin{cases} \frac{e^{-\frac{x}{y}} e^{-y}}{y} & x > 0, y > 0 \\ 0 & elsewhere \end{cases}$$

   a) Show that $f(x,y)$ is a density
   b) Find the conditional density of $X|Y$, then derive $E(X|Y)$ and $Var(X|Y)$
   c) Derive $E(X)$ and $Var(X)$

2) **(30 mins.)**

   a) If Y_1 and Y_2 are independent $N(0,1)$ prove that (Z_1, Z_2), where $Z_1 = Y_1$ and $Z_2 = Y_1 - Y_2$, is a Gaussian vector and (Z_1, Z_2) are not independent
   b) Now consider the gaussian vector $\mathbf{X} = (X_1, X_2, X_3)$ with zero mean and covariance matrix:

$$\begin{bmatrix} 1 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   b.1) Find the distribution of $X_1 + X_2$
   b.2) Calculate $E(X_2 + X_3 | X_3)$ and $Var(X_2 + X_3 | X_3)$

3) **(30 mins.)** Let X be $\Gamma(\alpha, \beta)$, where

$$f_X(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}, \quad 0 < x < \infty,$$

$\alpha > 0$, $\beta > 0$, the Gamma function is $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$, $E(X) = \frac{\alpha}{\beta}$, $Var(X) = \frac{\alpha}{\beta^2}$

   a) Find the method of moments estimators of α and β, denoted $\tilde{\alpha}$ and $\tilde{\beta}$
   b) Supposing that $\alpha = 1$, find the MLE estimator for $\tau = \frac{1}{\beta}$, denoted $\hat{\tau}$
   c) Is $\hat{\tau}$ the best estimator in the sense of the MSE?

4a) **(30 mins.)** Let X be a gaussian distribution $N(\mu, \sigma^2)$

a) Find the method of moments estimator for μ and σ^2. Are they unbiased?

b) Find the maximum likelihood estimator for μ supposing that σ^2 is given, equal to σ_0^2. Is it unbiased?

c) Find the maximum likelihood estimator for σ^2 supposing that μ is given, equal to μ_0. Is it unbiased?

4b) **(30 mins.)** Let $X_1, ...X_n$ be a random sample from a population $N(\mu, 1)$ and consider testing for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. (Note that $P(N(0,1) \leq 1.96) = 0.975$, $P(N(0,1) \leq 3.04) = 0.9988$, $P(N(0,1) \leq 0.88) = 0.8106$).

a) Find the LRT (likelihood ratio test) statistic for this problem

b) Define the rejection area for $n = 81$, for a test of level 0.05

c) Suppose $\mu_0 = 0$, so the set of hypotheses is $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. Find the power of the test for $\mu = 0.12$ (level 0.05)