EMJMD QEM 2020-2021
First year, first semester
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Final exam

1,5 hour

For information purposes, an estimated time is given for each part.

Q1 *(5 minutes)* We consider the following optimisation problem where f is convex, the equality constraints g_i are affine and the inequality constraints h_j are convex from U an open convex set of $\mathbb{R}^n$ to $\mathbb{R}$.

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } f(x) \\ g_i(x) = 0, \, \forall i = 1, \dots p \\ h_j(x) \leq 0, \, \forall j = 1 \dots, q \\ x \in U \end{array} \right.$$

What is the Slater condition for the constraint qualification?

Q2 *(5 minutes)* We consider a stationary dynamical optimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } \sum_{t=0}^{\infty} \beta^t f(a_t, s_t) \\ s_{t+1} = g(a_t, s_t), \quad t \geq 0 \\ (a_t, s_t) \in A, \, t \geq 0 \end{array} \right.$$

We denote by V the value function associated to this problem and we assume that it is well defined for all initial states s_0. What is the Bellman equation satisfied by V?

Exercise 1 *(45 minutes)* Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by:

$$f(x, y) = x^2 + xy + y^2 - 10x - 11y$$

1) Compute the gradient and the Hessian matrix of f at (x, y) and show that f is convex.

We now consider the following optimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Minimise } f(x, y) \\ 2x + y - 2 \leq 0 \end{array} \right.$$

2) Write the first order optimality condition and explain why they are also sufficient.

3) Find the unique point satisfying the first order condition and the associated multiplier.

We now consider the following problem where c is a given real number:

$$(\mathcal{Q}_c)\begin{cases}\text{Minimise } f(x,y)\\ 2x+y-2\leq 0\\ x+c\leq 0\end{cases}$$

4) Write the first order optimality conditions of the problem $(\mathcal{Q}_c)$ and explain why they are sufficient.

5) Show that if $c<1$, then the solution of the Problem $(\mathcal{P})$ is also a solution of the Problem $(\mathcal{Q}_c)$.

6) Using the conditions of Question 4, show that the point $(-c, 2+2c)$ is the solution of the Problem $\mathcal{Q}_c$ if $c \in [1, \frac{7}{3}]$ and that the point $(-c, \frac{11}{2}+\frac{c}{2})$ is the solution if $c > \frac{7}{3}$.

Exercise 2 *(35 minutes)* We consider a mine with a stock Q_0 of ore. The mine will be closed after three years of activities. The price of the ore is normalised equal to 1. Q_t is the stock at the beginning of the period t, q_t is the quantity extracted at period t, the cost of extraction is given by $q_t^2/2Q_t$.

The objective is to solve this problem **either** using the first order necessary condition **or** by the dynamical programming algorithm and backward induction.

1) Show that the problem to be solved is the following:

$$\begin{cases}\text{Maximise } \sum_{t=0}^{2} q_t - \frac{q_t^2}{2Q_t}\\ Q_{t+1}=Q_t-q_t,\ t=0,1,2\\ Q_3\geq 0\\ q_t\geq 0,\ t=0,1,2\end{cases}$$

2) Solve the above problem by **one** of the following two methods:

a) Write the first order necessary condition. Hint: do not forget the positivity constraint on Q_3.

b) By analysing the conditions concerning q_2, Q_2 and Q_3, show that the optimal solution satisfies $q_2^* = Q_2^*$, $Q_3^* = 0$.

c) Then, deduce from the optimal actions from the other equations.

d) What is the optimal value $V(Q_0)$.

or

a') Solve the problem using backward induction;

b') Write the solution path $(Q_0, q_0), (Q_1, q_1), (Q_2, q_2)$;

c') Compute the maximum value $V_0(Q_0)$.

3) Show that the value function $V(Q_0)$ is linear and compute its derivative.