University of Paris 1 Pantheon - Sorbonne 2018 - 2019
QEM + MMEF October 2019

Midterm exam of Probability and Statistics (1H)

No calculators, books, or notes may be used.

EX0 : 60%, EX1 : 16%, EX2 : 24%

Exercise 0 All the questions are independent

a) Remind **PRECISELY** the theoretical definition of a probability, the definition of a partition and the definition of a uniform random variable in the discrete case.

b) How many distinct sequences can we make using 3 letter "A"s and 5 letter "B"s ? (AAABBBBB, AABABBBB, etc.)

c) Thirty people are invited to a party. Each person accepts the invitation, independently of all others, with probability 1/3. Let X be the number of accepted invitations. Find $E[X]$.

d) Let X and Y be two independent Bernoulli random variables with the same distribution. Are $X + Y$ and $X - Y$ independent ?

e) Let X be a discrete random variable. Prove that for every $a \in \mathbb{R}$,

$$E((X - a)^2) = Var(X) + (E(X) - a)^2.$$

Deduce from this result the infimum of the mapping $a \to E((X - a)^2)$.

f) **[BONUS QUESTION]** Let $P : \mathcal{A} \to [0, 1]$ be a probability and $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$.

f_1) Prove that

$$P(\cup_{i=0}^{n} A_i) \leq \sum_{i=0}^{n} P(A_i).$$

Deduce that

$$P(\cup_{i=0}^{\infty} A_i) \leq \sum_{i=0}^{\infty} P(A_i).$$

f_2) If $\forall n \in \mathbb{N}$ $P(A_n) = 1$, prove that $P(\cap_{i=0}^{\infty} A_i) = 1$

Exercise 1 I have a bag with 3 coins in it. One of them is a fair coin, but the others are biased trick coins. When flipped, the three coins come up heads with probability $0.5, 0.6, 0.1$ respectively. Suppose that I pick one of these three coins uniformly at random and flip it three times.

(a) What is $P(\{H, T, T\})$? (That is, it comes up heads on the first flip and tails on the second and third flips.)

(b) Assumimg that the three flips, in order, are $\{H, T, T\}$, what is the probability that the coin that I picked was the fair coin ?

Exercise 2 We consider a sequence $(X_n)_{n\in\mathbb{N}^*}$ of independent random variables such that $\forall n \in \mathbb{N}^*$, $X_n \hookrightarrow \mathcal{P}(\lambda_n)$ where $\lambda_n \in \mathbb{R}_+$.

1) Compute (do the computation) $E[X_1]$.

2) Prove that $X_1 + X_2 \hookrightarrow \mathcal{P}(\lambda_1 + \lambda_2)$

3) [**BONUS QUESTION**] Prove that $\forall n \in \mathbb{N}^*$,

$$X_1 + ... + X_n \hookrightarrow \mathcal{P}(\lambda_1 + ... + \lambda_n).$$